

# TIME DOMAIN ANALYSIS OF IMPULSE RESPONSE TRAINS

THOMAS G. KINCAID

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TIME DOMAIN ANALYSIS OF IMPULSE RESPONSE TRAINS

Thomas G. Kincaid

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Abstract

An impulse response train is a signal that can be described as the response of a linear time-invariant system to a sequence of equally spaced impulses of varying areas. The impulse response associated with such a signal is called the kernel of the impulse response train.

A variety of physical systems generate signals in a manner indicating that the signals can be modeled by impulse response trains. Examples of such signals are the voiced sounds of speech, and the individual tones of wind instruments. Knowledge of the kernel of such physically generated signals would be useful for two reasons. First, the physical generating system can be simulated by a linear time-invariant system with the kernel as impulse response. Second, the class of signals generated by the system can be characterized by the kernel.

This report shows how to find the kernel of an impulse response train directly from the signal itself. The method assumes that the spacing of the impulses is known, but requires no knowledge of their areas, and also that the impulse response train is of finite duration. Since the kernel of an impulse response train is rarely unique, the method cannot always find the impulse response of the system that actually generated the signal. Rather, the method finds the kernel of shortest duration. For impulse response trains of finite duration there is only one such kernel, and all other kernels are impulse response trains having it as their kernel. Therefore, for the purposes of simulating the system and characterizing the signal, the kernel of minimum duration is sufficient.

The method used to find the kernel involves only matrix multiplication and the solving of simultaneous linear equations. Once the kernel is found, the impulse areas can be determined, again, by the solution of simultaneous linear equations. All of these operations can be routinely carried out by an electronic digital computer.

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## I. IMPULSE RESPONSE TRAINS

An impulse response train (IRT) is a signal that can be described as the response of a linear time-invariant system to a sequence of equally spaced impulses of varying areas. An example of an IRT is shown in Fig. 1. The impulse response associated

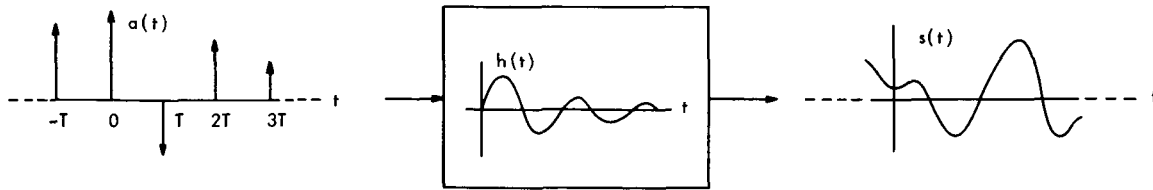


Fig. 1. The impulse response train  $s(t)$  is the response of the linear time-invariant system with impulse response  $h(t)$  to the sequence of equally spaced impulses of varying area  $a(t)$ .

with a given IRT will be called the kernel of the IRT. Thus, in Fig. 1,  $h(t)$  is the kernel of  $s(t)$ .

It is our objective to show how the kernel of a given IRT can be determined directly from the IRT itself, knowing only the spacing of the driving impulses, but nothing about their areas.

### 1.1 IMPULSE RESPONSE TRAINS AND PHYSICAL SYSTEMS

As motivation for finding the kernel of an IRT, we note that a variety of physical systems generate signals in a manner indicating that the signals can be well modeled by IRT. These physical systems generate signals by driving what essentially (over the time interval of interest) is a linear time-invariant system by regularly recurring pulses of variable amplitude, as shown in Fig. 2. Since these driving pulses can themselves be thought of as the output of a linear system driven by regularly spaced impulses, the generated signal can be modeled as the response of a composite linear time-invariant

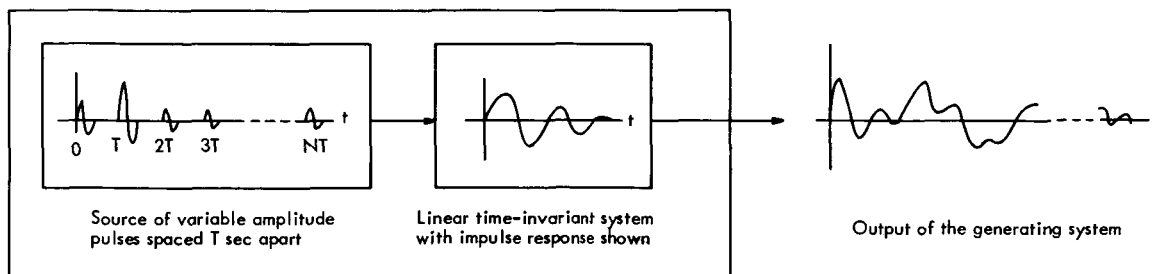


Fig. 2. A physical generating system that generates signals by driving a linear time-invariant system with a sequence of equally spaced impulses of varying amplitude.

system, as shown in Fig. 3. Examples of signals generated in this manner are the voiced sounds of speech,<sup>1</sup> and the individual tones of wind instruments.<sup>2</sup>

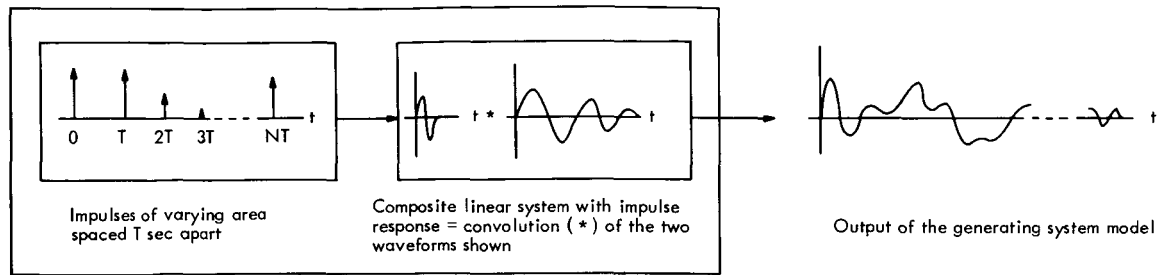


Fig. 3. Model of the physical generating system of Fig. 2. The output of this system is the same as that of Fig. 2. The impulse response of the composite linear system is the kernel of the output signal.

A knowledge of the impulse response of this composite linear system, which is then the kernel of the generated signal, is useful for two reasons. (i) It suggests a means of simulation of the physical system. The system could be simulated by a linear time-invariant system, with the kernel as impulse response, driven by impulses with the appropriate spacing and areas. In practice, of course, the impulses are approximated by short sharp pulses. Vowel sounds of speech have been successfully synthesized in this manner.<sup>3,4</sup> (ii) It is characteristic of the signals generated by the physical system. Since the generated signal is a linear combination of delayed versions of the kernel, the kernel is the fundamental "building block" of the signal. As an example of the utility of such a characterization, a knowledge of the kernel has been used to construct rejection filters for these types of signals.<sup>5</sup>

Given that it is worth while to find the kernel, how do we go about it? If successive impulse responses do not overlap, then there is no problem because the kernel is obvious by inspection, so we shall assume that this is not the case. When they do overlap, it might be possible to observe the waveform of the driving pulses within the actual physical system, and to make measurements that determine the impulse response of the linear system which they drive. Then the impulse response of the kernel can be determined by convolving these two. An alternative method would be to somehow find the kernel directly from the generated signal itself. In this approach it is assumed that the kernel somehow imposes a constraint on the generated signal, and that a knowledge of this constraint can be used to extract the waveform of the kernel. Such a constraint will be shown to exist and will be exploited to find the waveform of the kernel.

## 1.2 THE KERNEL OF MINIMUM DURATION

At this point it should be noted that the same IRT can have more than one kernel. For example, a situation could be visualized in which a particular IRT  $s(t)$  had a kernel  $h(t)$

which was in turn an IRT with the same spacing of impulses. Then it can be seen that the kernel of  $h(t)$  is also a kernel of  $s(t)$ . As another example, the entire IRT  $s(t)$  could be its own kernel, and  $s(t)$  could be synthesized by a linear system with impulse response  $s(t)$ , driven by one unit impulse! In fact, this example makes it clear that an IRT with just one kernel is of no interest, since that kernel would have to be the waveform itself.

Since it is possible for the same IRT to have several different kernels, then any method attempting to find the kernel directly from the waveform cannot be guaranteed to find the waveform of the composite impulse response that actually generated a given physical signal. Moreover, when the beginning and ending of an IRT are known, that is, when it is of finite duration, there is one kernel with particularly appealing properties. This is the kernel of shortest duration, or minimum kernel as we shall call it. These properties are the following. (i) The minimum kernel is unique. (ii) The set of impulses that go with the minimum kernel to synthesize the IRT are unique. (iii) Any other kernel of the IRT is an IRT with the minimum kernel as its kernel.

For the previously avowed purposes of simulating the generating system and characterizing the signal, the minimum kernel is clearly sufficient. Furthermore, since physically generated signals of the type that we are interested in are usually of finite duration, having to know the beginning and ending is not a serious restriction. Therefore, we shall concentrate our efforts on finding the minimum kernel of a given IRT.

The method that will be used to find the minimum kernel has as its basis the fact that sample pairs of an IRT taken one period apart are linearly dependent. This dependence will be demonstrated in Section II, and subsequently exploited to determine sample values of the kernel of the IRT. The entire procedure involves nothing more complicated than matrix multiplication and the solving of simultaneous linear equations. Once the minimum kernel is found, the impulse areas can be determined, again by the solution of simultaneous linear equations. All of these operations can be routinely carried out by an electronic digital computer.

The method does require knowledge of the spacing of the impulses. This condition is equivalent to knowing the pitch period for a voiced sound of speech, or the fundamental frequency of a musical tone. These can usually be determined by direct inspection or by frequency analysis of the signal (see Section VI).

### 1.3 DEFINITIONS AND TERMINOLOGY

It is desirable to have some names by which to refer to the parameters peculiar to IRT. The following terminology will be used. Let

$$s(t) = \sum_{n=N_1}^{N_2} a_n h(t-(n-1)T),$$

where

$N_1$  and  $N_2$  are integers

$T$  is a real positive number

$h(t)$  is a real bounded time function of duration  $<DT$ ,  $D$  a positive integer

$\{a_n\}$  is a set of real numbers.

Then

- (i)  $s(t)$  is an impulse response train.
- (ii)  $T$  is the period of  $s(t)$ . (Note that the use of the term "period" does not imply that  $s(t)$  is periodic.)
- (iii)  $h(t)$  is the kernel of the IRT  $s(t)$ .
- (iv)  $D$  is the span of the kernel  $h(t)$ , that is, the number of periods "spanned" by the duration of the kernel.
- (v) the  $\{a_n\}$  are the impulse areas.



## II. HOMOGENEOUS EQUATIONS RELATING IMPULSE RESPONSE TRAIN SAMPLES

The characteristic feature of an IRT

$$s(t) = \sum_{n=N_1}^{N_2} a_n h(t-(n-1)T)$$

is that it is a sum of equally spaced signals  $\{a_n h(t-(n-1)T)\}$ , each of which has the same wave shape as the kernel  $h(t)$ . If the duration of  $h(t)$  is less than or equal to the period  $T$ , the wave shape of  $h(t)$  will be obvious by inspection. If, however, the duration of  $h(t)$  is longer than  $T$ , then the signals will overlap, and their waveform can no longer be determined by direct inspection of  $s(t)$ .

Nevertheless, it might be suspected that the wave shape of  $s(t)$  is somehow constrained by the fact that it is a sum of equally spaced signals of the same wave shape.

We shall now show that such a constraint does indeed exist. In particular, it will be shown that appropriately chosen samples of the waveform of  $s(t)$  are alternately signed coefficients of a set of linear homogeneous equations with samples of the waveform of  $h(t)$  as solutions.

### 2.1 HOMOGENEOUS EQUATIONS FOR A PARTICULAR CASE

It is convenient first to demonstrate the existence of this relation for a particular example. Let us construct an IRT  $s(t)$  with the kernel shown in Fig. 4. For convenience, the time origin is chosen at the beginning of the kernel waveform. Let  $s(t)$  be the IRT

$$s(t) = \sum_{n=N_1}^{N_2} a_n h(t-(n-1)T),$$

a portion of which is shown in Fig. 5. The figure also shows the waveforms of the  $\{a_n h(t-(n-1)T)\}$  plotted on separate time axes. Note that  $h(t)$  has span 2 (see definitions in sec. 1.3). To show that there exists a relation between the samples of  $s(t)$  and the kernel  $h(t)$ , proceed as follows (see Fig. 6).

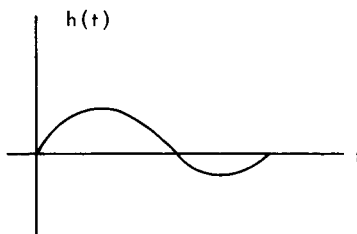


Fig. 4. The kernel  $h(t)$ .

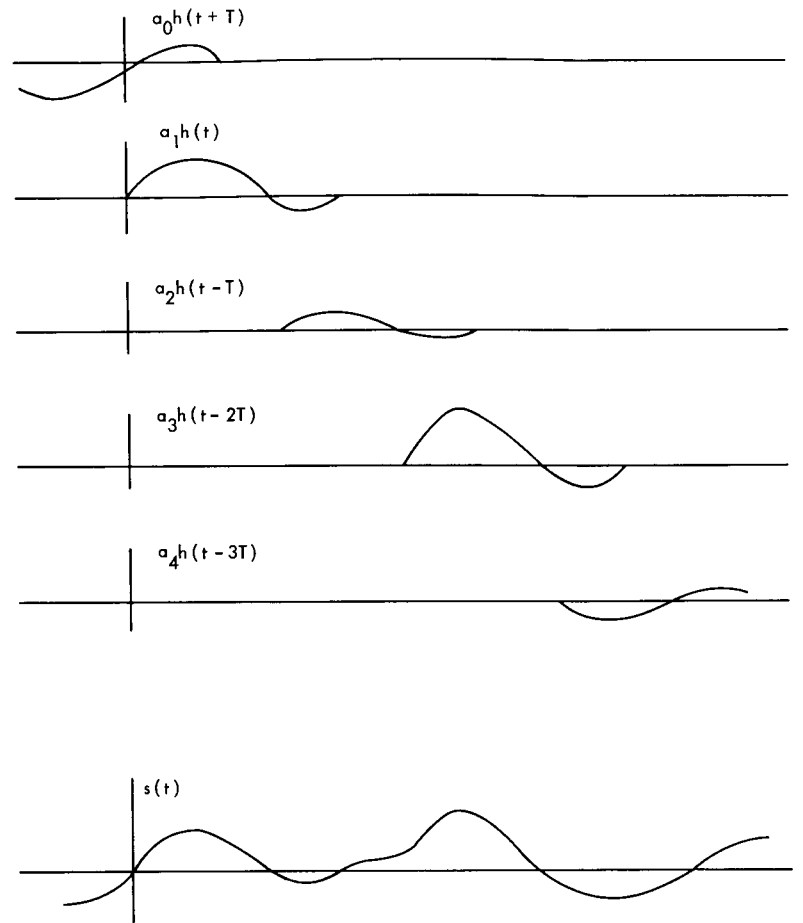


Fig. 5. Individual waveforms  $\{a_n^h(t-(n-1)T)\}$  and their sum  $s(t)$ .

Starting at the time origin, and working in both directions, divide the time axis into  $T$ -long intervals, called periods. Number these periods 1, 2, 3, ... in the direction of increasing time, and 0, -1, -2, ... in the direction of decreasing time. Note that this causes the signal  $a_p^h(t-(p-1)T)$  to begin in the  $p^{\text{th}}$  period.

Now, arbitrarily select one sample of  $s(t)$  from each period in such a way that selected samples are spaced  $T$  seconds apart. Call any set of samples of  $s(t)$  chosen in this way a set of periodic samples of  $s(t)$ . Label the sample from the first period  $s_{1,1}$ , from the second period  $s_{2,1}$ , etc. Now arbitrarily select a second set of periodic samples of  $s(t)$ , different from the first. Label the sample of this set from the first period  $s_{1,2}$ ,  $s_{2,2}$ , etc. In both sets then, the subscript scheme is  $s_{\text{period}, \text{sample in period}}$ . These two sets of periodic sample pairs are shown in Fig. 7. Call two such sets of periodic samples a set of periodic sample pairs of  $s(t)$ .

Consider now the samples of a particular waveform  $a_p^h(t-(p-1)T)$  taken at the same times as the set of periodic sample pairs of  $s(t)$  selected above. Call these the coincident samples of  $a_p^h(t-(p-1)T)$ . Label the two coincident samples of  $a_p^h(t-(p-1)T)$  in the  $p^{\text{th}}$  period (that is, the period in which  $a_p^h(t-(p-1)T)$  begins)  $a_{p,1,1}$  and  $a_{p,1,2}$ ,

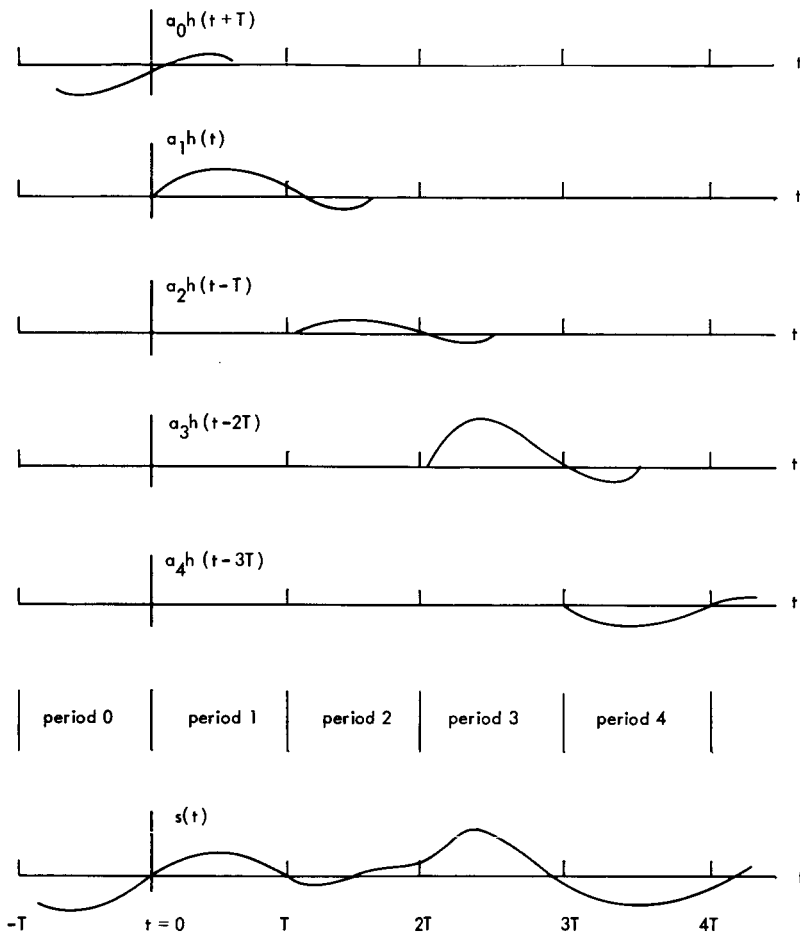


Fig. 6. Division of the time axis into numbered periods.

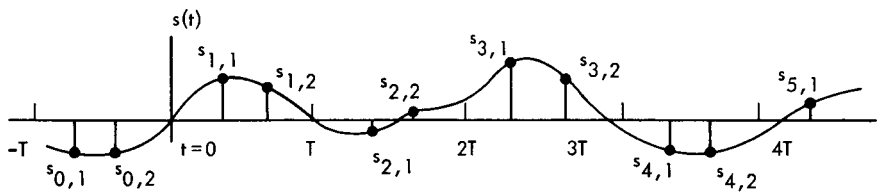


Fig. 7. A set of periodic sample pairs of  $s(t)$ .

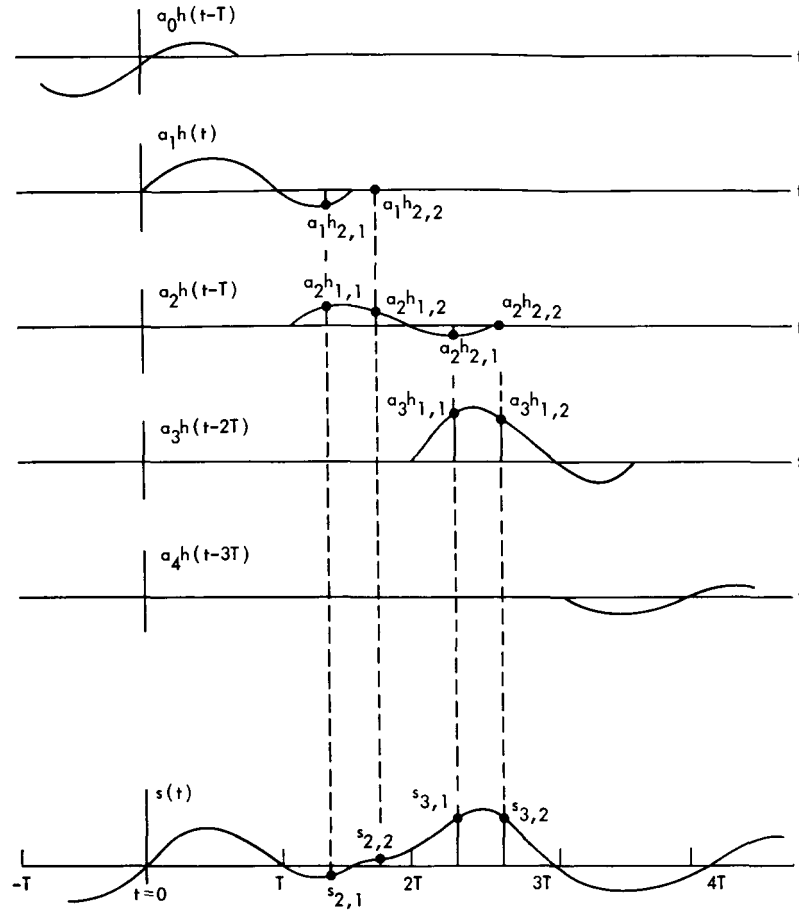


Fig. 8. Coincident samples of the  $\{a_n h(t-(n-1)T)\}$  for two periods of the chosen set of periodic sample pairs of  $s(t)$ .

respectively. Label the two coincident samples of  $a_p h(t-(p-1)T)$  in the  $(p+1)^{\text{th}}$  period  $a_{p2,1}$  and  $a_{p2,2}$ , respectively. Since in our example  $h(t)$  has span 2,  $a_p h(t-(p-1)T)$  will be zero in all the other periods. Figure 8 shows the coincident samples of the  $\{a_p h(t-(p-1)T)\}$  for two periods of the chosen set of periodic sample pairs of  $s(t)$ , and how these coincident samples are labeled.

Now, consider the selected samples of  $s(t)$  from any two consecutive periods, say the second and third. These samples of  $s(t)$  can then be written as a linear combination of samples of  $h(t)$  as follows.

$$s_{2,1} = a_{12,1} + a_{21,1}$$

$$s_{2,2} = a_{12,2} + a_{21,2}$$

$$s_{3,1} = a_{22,1} + a_{31,1}$$

$$s_{3,2} = a_{22,2} + a_{31,2}$$

(1)

Perform the following operations on these equations.

- (i) Multiply the equations in ascending order of subscripts of  $s_{i,j}$  by the "coincident samples" of  $h(t)$  in descending order of subscripts.
- (ii) Multiply alternate equations by  $-1$
- (iii) Add the equations.

Carrying out these operations yields

$$\begin{aligned} s_{2,1}h_{2,2} &= a_1h_{2,1}h_{2,2} + a_2h_{1,1}h_{2,2} \\ -s_{2,2}h_{2,1} &= -a_1h_{2,2}h_{2,1} - a_2h_{1,2}h_{2,1} \end{aligned} \quad (2)$$

$$\begin{aligned} s_{3,1}h_{2,1} &= a_2h_{2,1}h_{1,2} + a_3h_{1,1}h_{1,2} \\ -s_{3,2}h_{1,1} &= -a_2h_{2,2}h_{1,1} - a_3h_{1,2}h_{1,1} \\ \hline s_{2,1}h_{2,2} - s_{2,2}h_{2,1} + s_{3,1}h_{1,2} - s_{3,2}h_{1,1} &= 0. \end{aligned} \quad (3)$$

Equation 3 is a homogeneous equation relating the samples of the IRT to the coincident samples of the kernel. Since the method by which Eq. 3 was obtained does not depend upon which two consecutive periods the samples of  $s(t)$  are chosen from, similar equations hold for sample pairs from every two consecutive periods. That is, the samples  $h_{1,1}$ ,  $h_{1,2}$ ,  $h_{2,1}$ , and  $h_{2,2}$  of  $h(t)$  must satisfy the set of homogeneous equations

$$\begin{aligned} & \cdot \quad \cdot \quad \cdot \quad - \quad - \quad - \quad - \quad - \quad - \quad \cdot \\ s_{1,1}h_{2,2} - s_{1,2}h_{2,1} + s_{2,1}h_{1,2} - s_{2,2}h_{1,1} &= 0 \\ s_{2,1}h_{2,2} - s_{2,2}h_{2,1} + s_{3,1}h_{1,2} - s_{3,2}h_{1,1} &= 0 \\ s_{3,1}h_{2,2} - s_{3,2}h_{2,1} + s_{4,1}h_{1,2} - s_{4,2}h_{1,1} &= 0 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ & \text{etc.} \end{aligned}$$

Or, more compactly

$$\{s_{p,1}h_{2,2} - s_{p,2}h_{2,1} + s_{p+1,1}h_{1,2} - s_{p+1,2}h_{1,1} = 0\}. \quad (4)$$

These equations show that the periodic sample pairs of  $s(t)$  are linearly dependent, and that their dependence is determined by the "coincident" samples of the kernel  $h(t)$ . Another way of looking at these equations is to note that a necessary condition for  $h(t)$  to be a kernel of  $s(t)$  is that the coincident samples of  $h(t)$  satisfy the homogeneous equations written for the correct span.

Equations 4 are a special case of a more general result, which shows that similar homogeneous equations of order  $2D$  exists for IRT  $s(t)$  having kernels of span  $D$ .

For example, for  $D = 1$

$$\begin{array}{c}
- - - - \\
s_{1,1}h_{1,2} - s_{1,2}h_{1,1} = 0 \\
s_{2,1}h_{1,2} - s_{2,2}h_{1,1} = 0 \\
s_{3,1}h_{1,2} - s_{3,2}h_{1,1} = 0 \\
- - \\
- -
\end{array}$$

for  $D = 2$  we have the equations just derived, for  $D = 3$ ,

$$\begin{array}{c}
- - - - - \\
s_{1,1}h_{3,2} - s_{1,2}h_{3,1} + s_{2,1}h_{2,2} - s_{2,2}h_{2,1} + s_{3,1}h_{1,2} - s_{3,2}h_{1,1} = 0 \\
s_{2,1}h_{3,2} - s_{2,2}h_{3,1} + s_{3,1}h_{2,2} - s_{3,2}h_{2,1} + s_{4,1}h_{1,2} - s_{4,2}h_{1,1} = 0 \\
s_{3,1}h_{3,2} - s_{3,2}h_{3,1} + s_{4,1}h_{2,2} - s_{4,2}h_{2,1} + s_{5,1}h_{1,2} - s_{5,2}h_{1,1} = 0 \\
- - - - - \\
- - - - -
\end{array}$$

and, in general, for an  $s(t)$  with a kernel  $h(t)$  of span  $D$

$$\left\{ \sum_{j=0}^{D-1} s_{p+j,1}h_{D-j,2} - s_{p+j,2}h_{D-j,1} \right\} = 0,$$

where  $p$  is the integer assigned to the first of any  $D$  consecutive periods of  $s(t)$ . This general result is proved in section 2.2. This proof may be omitted without breaking the continuity of the presentation.

## 2.2 HOMOGENEOUS EQUATIONS FOR THE GENERAL CASE

Equations of the form of Eqs. 4 hold for the general case of IRT  $s(t)$  of period  $T$ , with kernel  $h(t)$  of span  $D$ . The proof follows.

As in Fig. 6, divide  $s(t)$  into numbered periods of length  $T$ . Starting with the  $p^{\text{th}}$  period, choose a set of periodic sample pairs from  $D$  consecutive periods of  $s(t)$ . These samples of  $s(t)$  can be written as sums of the coincident samples of the waveforms  $\{a_n h(t-(n-1)T)\}$ . Using the notation established in the previous section, we find that these equations are

$$\left\{ \begin{array}{l} s_{p+j,1} = \sum_{i=1}^D s_{p+j-i}h_{i,1} \\ s_{p+j,2} = \sum_{i=1}^D a_{p+j-i}h_{i,2} \end{array} \right\}_{j=0}^{D-1}. \quad (5)$$

Equations 1 is a special case of these equations.

Multiplying the equation for the first sample of the  $(p+j)^{\text{th}}$  period by  $h_{D-j, 2}$ , and that for the second sample by  $-h_{D-j, 1}$ ,

$$\left\{ \begin{array}{l} s_{p+j, 1} h_{D-j, 2} = \sum_{i=1}^D a_{p+j-i} h_{i, 1} h_{D-j, 2} \\ -s_{p+j, 2} h_{D-j, 1} = - \sum_{i=1}^D a_{p+j-i} h_{i, 2} h_{D-j, 1} \end{array} \right\}_{j=0}^{D-1} \quad (6)$$

Equations 2 is a special case of these equations. Summing these equations on the index  $j$  gives the equation

$$\sum_{j=0}^{D-1} s_{p+j, 1} h_{D-j, 2} - s_{p+j, 2} h_{D-j, 1} = \sum_{j=0}^{D-1} \sum_{i=1}^D a_{p+j-i} (h_{i, 1} h_{D-j, 2} - h_{i, 2} h_{D-j, 1}). \quad (7)$$

The right side of this equation is shown to be zero as follows. First, consider the term for which  $i = m$  and  $j = n$ . As  $n$  takes on the values of  $j$  from 0 to  $D - 1$ , note that  $D - n$  takes on each of the values from  $D$  to 1 once and only once, i.e., all values in the range of  $i$ . Similarly, as  $m$  takes on the values of  $i$  from 1 to  $D$ ,  $D - m$  takes on each of the values from  $D - 1$  to 0 once and only once, that is, all values in the range of  $j$ . Hence, for the term for which  $i = m$  and  $j = n$ , there exists one and only one other term in the summation for which  $i = D - n$  and  $j = D - m$ . This term is

$$a_{p+n-m} (h_{D-n, 1} h_{m, 2} - h_{D-n, 2} h_{m, 1})$$

which is the negative of the term for which  $i = m$  and  $j = n$ . Thus the terms of the summation on the right side of Eq. 7 cancel in pairs, and the sum is zero.

Thus the set of sample pairs  $\{h_{i, 1}, h_{i, 2}\}_{i=1}^D$  of  $h(t)$  satisfies the set of homogeneous equations

$$\left\{ \sum_{j=0}^{D-1} s_{p+j, 1} h_{D-j, 2} - s_{p+j, 2} h_{D-j, 1} = 0 \right\}, \quad (8)$$

where  $p$  is the integer assigned to the first of any  $D$  consecutive periods of  $s(t)$ . Equations 4 is a special case of these equations.

By a further generalization of the above derivation using methods similar to that discussed at the end of the previous section, it is possible to select any even number  $k$  of samples per period and write corresponding homogeneous equations of order  $Dk$ . It is sufficient for the purposes of this investigation to consider only two samples per period, because of the difficulties inherent in solving simultaneous linear equations in many unknowns.

### III. PROPERTIES OF THE SOLUTIONS TO THE HOMOGENEOUS EQUATIONS

It has been shown that any set of periodic sample pairs of an IRT are alternately signed coefficients of a set of linear homogeneous equations, with coincident samples of the kernel as solutions (Eqs. 8). Thus, for any set of periodic sample pairs of an arbitrary waveform  $s(t)$ , a necessary condition for a set of numbers  $\{h_{ij}\}$  to be the coincident samples of a kernel of span  $D$  is that these numbers solve the appropriate homogeneous equations of order  $2D$ . This result suggests that we should attempt to find samples of the unknown minimum kernel of an IRT by selecting many sets of periodic sample pairs of the IRT, and solving the appropriate sets of homogeneous equations to find the coincident samples of the kernel. This approach is, in principle, the one that will be used. There are problems, however, which prevent its direct application.

(i) Solving the homogeneous equations is not a sufficient condition for a set of numbers to be the coincident samples of a kernel. Just because a set of numbers will solve the homogeneous equations does not guarantee them to be coincident samples of a kernel. For example, the trivial (zero) solution solves every set of homogeneous equations, but is certainly not the solution we seek. Our first objective, then, will be to find a sufficient condition for a solution to the homogeneous equations to be the coincident samples of a kernel.

(ii) For the cases of interest, the solution will never be unique. This follows from the fact that whenever the homogeneous equations have a unique solution, it is always the trivial solution, which is of no interest. Otherwise, the equations have a  $P$ -parameter infinity of solutions, and we are faced with the problem of selecting one from this infinity of solutions. Rather than trying to resolve this problem in detail, we shall focus our attention on one particular nontrivial solution, the 1-parameter solution. It will be shown that for IRT of finite duration the coincident samples of the minimum kernel are always found as 1-parameter solutions to the set of homogeneous equations written for the proper span.

(iii) The span  $D$  for which the homogeneous equations are to be written is not known. Since we are going to consider IRT of finite duration, we can assume that the kernel is of finite duration, but we have no knowledge of just how long a time it lasts. In connection with our study of 1-parameter solutions, we shall discover a method of finding the span of the minimum kernel which is always applicable when the beginning and ending of the IRT are known.

We shall confine ourselves here and in Section IV to an investigation of the properties of the solutions to the homogeneous equations for just one set of periodic sample pairs, that is, we shall try to find just two samples of the kernel per period. Later we shall use our results to determine as many samples of the kernel as we desire. For the present, we shall consider methods of solving the problems posed above for just one set of periodic sample pairs.



### 3.1 ACCEPTABLE AND UNACCEPTABLE SOLUTIONS

Suppose that a set of periodic sample pairs is selected from some IRT  $s(t)$ , and the corresponding homogeneous equations are solved. Since the equations are homogeneous, there will always be a solution. The solution may be the trivial one, a 1-parameter solution, a 2-parameter solution, and so on. The problem is to determine whether any of the  $P$ -parameter infinity of solutions can be coincident samples of a kernel of  $s(t)$ .

In any case, it is clear that the solutions can always be divided into two mutually exclusive categories.

(i) Those solutions that are coincident samples of a waveform that can be a kernel of  $s(t)$ . Call these acceptable solutions.

(ii) Those solutions that are not coincident samples of a waveform that can be a kernel of  $s(t)$ . Call these unacceptable solutions.

The definitions of acceptable and unacceptable solutions need some amplification if they are to be used to test any particular solution  $\{h_{ij}\}$  to a set of homogeneous equations. Specifically, a solution is acceptable if the following equations, hereafter called the generating equations of  $s(t)$ ,

$$\begin{cases} s_{p,1} = a_{p-D+1}h_{D,1} + a_{p-D+2}h_{D,1} + \dots + a_p h_{1,1} \\ s_{p,2} = a_{p-D+1}h_{D,2} + a_{p-D+2}h_{D,2} + \dots + a_p h_{1,2} \end{cases} \quad (9)$$

can be solved for the impulse areas  $\{a_i\}$ . Equations 1 give an example of these equations. This criterion obviously guarantees the  $\{h_{ij}\}$  to be an acceptable solution by actually showing how the periodic sample pairs of  $s(t)$  can be reconstructed. If the generating equations cannot be solved for the  $\{a_i\}$ , then the solution  $\{h_{ij}\}$  is unacceptable.

As examples of acceptable and unacceptable solutions, consider an  $s(t)$  which is an IRT whose kernel has span  $D$ . The existence of at least one acceptable solution is guaranteed, namely, the coincident samples of the kernel. The existence of at least one unacceptable solution is also guaranteed, the trivial (zero) solution. It is of interest to note that for any waveform of finite duration, there is some span for which the homogeneous equations have an acceptable solution. This result is more obvious than might be supposed, since it follows readily from the observation that any waveform of finite duration can be considered to be an IRT merely by letting the waveform itself be the impulse response (kernel), and using one impulse. Then the homogeneous equation written for a span which includes the whole waveform will have solutions that are the same periodic pairs used as the coefficients of the equation.

Since this result is necessary later, we shall formalize it in a theorem.

**THEOREM 1.** Let  $\{s_{1,1}, s_{1,2}, \dots, s_{N,1}, s_{N,2}\}$  be any set of  $N$  consecutive periodic sample pairs from any waveform  $s(t)$  for some period  $T$ . Then there is at least one span for which the homogeneous equations have an acceptable solution.

Proof: The homogeneous equation for span N is

$$\sum_{n=1}^N s_{n,1} h_{N+1-n,2} - s_{n,2} h_{N+1-n,1} = 0.$$

This equation has the acceptable solution

$$h_{j,k} = s_{j,k};$$

that is, the samples of the IRT itself. This is a solution because the terms of the equation cancel in pairs, giving zero. It is acceptable because it can generate the original samples with impulse areas  $a_1 = 1$  and  $a_n = 0$ ,  $1 < n \leq N$ . Thus the homogeneous equation written for span N always has an acceptable solution. // (// denotes the end of a proof, / part of a proof.)

### 3.2 A SUFFICIENT CONDITION FOR AN ACCEPTABLE SOLUTION

Before proceeding with the theorem, which gives a sufficient condition for a solution to be acceptable, it is convenient to introduce some matrix notation. Let  $s(t)$  be an arbitrary waveform of N periods numbered from 1 to N, increasing in the direction of increasing time. For any set of periodic sample pairs, the homogeneous equations of order 2D are

$$\begin{aligned} s_{1,1} h_{D,2} - s_{1,2} h_{D,1} + \dots + s_{D,1} h_{1,2} - s_{D,2} h_{1,1} &= 0 \\ s_{2,1} h_{D,2} - s_{2,2} h_{D,1} + \dots + s_{D+1,1} h_{1,2} - s_{D+2,2} h_{1,1} &= 0 \\ \vdots & \\ s_{N-D+1,1} h_{D,2} - s_{N-D+1,2} h_{D,1} + \dots + s_{N,1} h_{1,2} - s_{N,2} h_{1,1} &= 0. \end{aligned} \quad (10)$$

This set of equations will be given the matrix notation

$$\begin{bmatrix} S_D^N \\ H_D \end{bmatrix} = 0, \quad (11)$$

where

$$\begin{bmatrix} S_D^N \end{bmatrix} = \begin{bmatrix} s_{1,1} & -s_{1,2} & \dots & s_{D,1} & -s_{D,2} \\ s_{2,1} & -s_{2,2} & \dots & s_{D+1,1} & -s_{D+1,2} \\ \vdots & & & & \\ s_{N-D+1,1} & -s_{N-D+1,2} & \dots & s_{N,1} & -s_{N,2} \end{bmatrix}$$

$$H_D] = \begin{bmatrix} h_{D,2} \\ h_{D,1} \\ \vdots \\ h_{1,2} \\ h_{1,1} \end{bmatrix}$$

and 0] is the column zero matrix having the appropriate number of entries. Note that N and D do not refer to the number of rows and columns in any of the matrices, but the number of rows and columns can be computed from these numbers.

It is convenient to establish the following notation also. If s(t) is an IRT having a kernel h(t) of span D, the periodic sample pairs of s(t) can be written in terms of the coincident periodic sample pairs of h(t) by the following generating equations:

$$\begin{aligned} s_{1,1} &= a_{-D+2}h_{D,1} + a_{-D+3}h_{D-1,1} + \dots + a_1h_{1,1} \\ s_{1,2} &= a_{-D+2}h_{D,2} + a_{-D+3}h_{D-1,2} + \dots + s_1h_{1,2} \\ &\vdots \\ s_{N,1} &= a_{N-D+1}h_{D,1} + a_{N-D+2}h_{D-1,1} + \dots + a_Nh_{1,1} \\ s_{N,2} &= a_{N-D+1}h_{D,2} + a_{N-D+2}h_{D-1,2} + \dots + s_Nh_{1,2}. \end{aligned} \tag{12}$$

This set of generating equations can be written in matrix notation.

$$S^N] = \begin{bmatrix} H_D^N \\ A_D^N \end{bmatrix}, \tag{13}$$

where

$$S^N] = \begin{bmatrix} s_{1,1} \\ s_{1,2} \\ \vdots \\ s_{N,1} \\ s_{N,2} \end{bmatrix}$$

$$\begin{bmatrix} H_D^N \end{bmatrix} = \begin{bmatrix} h_{D,1}h_{D-1,1} & \cdot & \cdot & \cdot & \cdot & h_{1,1} & 0 & 0 & 0 & \cdot & \cdot \\ h_{D,2}h_{D-1,2} & \cdot & \cdot & \cdot & \cdot & h_{1,2} & 0 & 0 & 0 & \cdot & \cdot \\ 0 & h_{D,1} & h_{D-1,1} & \cdot & \cdot & \cdot & h_{1,1}^0 & 0 & \cdot & \cdot & \cdot \\ 0 & h_{D,2} & h_{D-1,2} & \cdot & \cdot & \cdot & h_{1,2}^0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & h_{D,1} & \cdot & \cdot & \cdot & \cdot & h_{1,1}^0 & \cdot & \cdot & \cdot \\ 0 & 0 & h_{D,2} & \cdot & \cdot & \cdot & \cdot & h_{1,2}^0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \text{etc.} & & & & & \\ \cdot & \cdot & \cdot & & & & & & & & \\ \cdot & \cdot & \cdot & & & & & & & & \end{bmatrix}$$

and

$$A_D^N = \begin{bmatrix} a_{-D+2} \\ \vdots \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

Finally, it is convenient to write a notation for the  $2(D-1)^{\text{th}}$ -order determinant of the matrix  $[H_D^{D-1}]$ . Denote this determinant by  $|H_D|$ . For example, for  $D=3$  this determinant is

$$|H_3| = \begin{vmatrix} h_{3,1} & h_{2,1} & h_{1,1} & 0 \\ h_{3,2} & h_{2,2} & h_{1,2} & 0 \\ 0 & h_{3,1} & h_{2,1} & h_{1,1} \\ 0 & h_{3,2} & h_{2,2} & h_{1,2} \end{vmatrix}$$

It will now be shown that a sufficient condition for an acceptable solution to the homogeneous equations is that  $|H_D| \neq 0$ .

**THEOREM 2.** Let  $\{s_{1,1}, s_{1,2}, \dots, s_{N,1}, s_{N,2}\}$  be any set of  $N$  consecutive sample pairs from a waveform  $s(t)$  for some period  $T$ . Let  $\{h_{D,2}, h_{D,1}, \dots, h_{1,2}, h_{1,1}\}$  be a solution to the set of homogeneous equations  $[S_D^N] H_D = 0$  such that  $|H_D| \neq 0$ . Then there exists a unique solution to the generating equations  $S^N = [H_D^N] A_D^N$ .

Proof: The theorem will first be proved for  $D=3$  and  $N=5$  to illustrate the procedure to be used in the general proof.

The equations for which a unique solution allegedly exists are  $S^5 = [H_3^5] A_3^5$  or

$$\begin{aligned} 1. \quad s_{1,1} &= a_{-1}h_{3,1} + a_0h_{2,1} + a_1h_{1,1} \\ 2. \quad s_{1,2} &= a_{-1}h_{3,2} + a_0h_{2,2} + a_1h_{1,2} \\ 3. \quad s_{2,1} &= a_0h_{3,1} + a_1h_{2,1} + a_2h_{1,1} \\ 4. \quad s_{2,2} &= a_0h_{3,2} + a_1h_{2,2} + a_2h_{1,2} \\ 5. \quad s_{3,1} &= a_1h_{3,1} + a_2h_{2,1} + a_3h_{1,1} \\ 6. \quad s_{3,2} &= a_1h_{3,2} + a_2h_{2,2} + a_3h_{1,2} \\ 7. \quad s_{4,1} &= a_2h_{3,1} + a_3h_{2,1} + a_4h_{1,1} \\ 8. \quad s_{4,2} &= a_2h_{3,2} + a_3h_{2,2} + a_4h_{1,2} \\ 9. \quad s_{5,1} &= a_3h_{3,1} + a_4h_{2,1} + a_5h_{1,1} \\ 10. \quad s_{5,2} &= a_3h_{3,2} + a_4h_{2,2} + a_5h_{1,2} \end{aligned}$$

Now, one of  $h_{1,1}$ ,  $h_{1,2}$  is not zero. For, if both are zero, the last column of  $|H_3|$  is zero, hence  $|H_3| = 0$ . But this is not so by assumption. Assume that  $h_{1,1}$  is not zero. Then the given equations  $[S_3^5] H_3 = 0$ , written out,

$$\begin{aligned} s_{1,1}h_{3,2} - s_{1,2}h_{3,1} + s_{2,1}h_{2,2} - s_{2,2}h_{2,1} + s_{3,1}h_{1,2} - s_{3,2}h_{1,1} &= 0 \\ s_{2,1}h_{3,2} - s_{2,2}h_{3,1} + s_{3,1}h_{2,2} - s_{3,2}h_{2,1} + s_{4,1}h_{1,2} - s_{4,2}h_{1,1} &= 0 \\ s_{3,1}h_{3,2} - s_{3,2}h_{3,1} + s_{4,1}h_{2,2} - s_{4,2}h_{2,1} + s_{5,1}h_{1,2} - s_{5,2}h_{1,1} &= 0, \end{aligned}$$

imply that the 10<sup>th</sup>, 8<sup>th</sup>, and 6<sup>th</sup> equations of  $S^5] = [H_3^5] A_3^5$  are linearly dependent upon the five equations that immediately precede them. For, let equations 5-10 be multiplied by  $h_{3,2}$ ,  $-h_{3,1}$ ,  $h_{2,2}$ ,  $-h_{2,1}$ ,  $h_{1,2}$ ,  $-h_{1,1}$ , respectively. The left-hand side is zero, by the third homogeneous equation above. The right-hand side is zero for reasons discussed in section 2.3 in connection with the derivation of the homogeneous equations. Thus the 10<sup>th</sup> equation is linearly dependent upon the previous five equations. Equations 8 and 6 are similarly dependent upon the five equations that precede them. Thus, the equations remaining to be solved are

$$\begin{aligned} 1. \quad s_{1,1} &= a_{-1}h_{3,1} + a_0h_{2,1} + a_1h_{1,1} \\ 2. \quad s_{1,2} &= a_{-1}h_{3,2} + a_0h_{2,2} + a_1h_{1,2} \\ 3. \quad s_{2,1} &= a_0h_{3,1} + a_1h_{2,1} + a_2h_{1,1} \\ 4. \quad s_{2,2} &= a_0h_{3,2} + a_1h_{2,2} + a_2h_{1,2} \\ 5. \quad s_{3,1} &= a_1h_{3,1} + a_2h_{2,1} + a_3h_{1,1} \\ 7. \quad s_{4,1} &= a_2h_{3,1} + a_3h_{2,1} + a_4h_{1,1} \\ 9. \quad s_{5,1} &= a_3h_{3,1} + a_4h_{2,1} + a_5h_{1,1}. \end{aligned}$$

The determinant of the coefficient matrix for this set of equations is  $(h_{1,1})^3 |H_3|$ , which cannot be zero, by assumption. Since there are seven equations in seven unknowns, and the coefficient matrix has a nonzero determinant, the solution exists and is unique.

If  $h_{1,1} = 0$ , then  $h_{1,2}$  cannot be zero. Equations 5, 7, and 9 can then be shown to be dependent on the others, and a similar proof carried out to show that a unique solution exists. /

The general proof follows along the same lines. The equations for which a unique solution allegedly exists are  $S^N] = [H_D^N] A_D^N$ . Now, one of  $h_{1,1}$ ,  $h_{1,2}$  is not zero. For, if both are zero, the last column of  $|H_D|$  is zero, hence  $|H_D| = 0$ . But this contradicts an assumption of the theorem.

Assume that  $h_{1,1}$  is not zero. Then the  $N-D+1$  homogeneous equations  $[S_D^N] H_D = 0$  imply that the  $2N^{\text{th}}$ ,  $2(N-1)^{\text{th}}$ , ...,  $2(N-D+1)^{\text{th}}$  equations of  $S_D^N] = [H_D^N] A_D^N$  are

linearly dependent upon the  $(2D-1)$  equations immediately preceding them. Therefore, these equations may be removed from consideration, since they are implied by the others. This leaves  $2N - (N-D+1) = N + D - 1$  equations in  $N + D - 1$  unknowns, that is, the first  $2(D-1)$  of the original equations remain, along with the first of each of the remaining pairs of equations. The determinant of the coefficient matrix is  $(h_{1,1})^{N-D+1} |H_D|$ , which cannot be zero, by assumption. Hence, the  $N + D - 1$  equations in  $N + D - 1$  unknowns have a solution, and it is unique. This solution also satisfies the remaining equations, since they were linearly dependent upon the equations having this solution.

If  $h_{1,1} = 0$ , then  $h_{1,2}$  cannot be zero. Then the first equation of each of the last  $N - D + 1$  pairs can be eliminated, and the proof proceeds as before, with the determinant of the coefficient matrix being  $(h_{1,2})^{N-D+1} |H_D|$ . //

### 3.3 1-PARAMETER ACCEPTABLE SOLUTIONS

As we have previously noted, one question that arises in the solution of linear equations is that of the uniqueness of the solution. It may be that the solution is unique, or there may be a 1-parameter, 2-parameter, etc. infinity of solutions. These possibilities present special problems in our case, for which the equations are homogeneous, and therefore the only unique solution is the trivial one, which is useless as the samples of a kernel. Therefore, we must always expect to be presented with an infinity of solutions in any case of interest.

Of the possible solutions of interest, the simplest that we can expect are the 1-parameter solutions. Not only are these the simplest, but they are easiest to interpret. The meaning of a 1-parameter solution is that every possible solution is just a constant times any other possible solution; that is, all of the solutions are scaled versions of each other. Thus if any one of the 1-parameter solutions is acceptable, they all must be (except, of course, the zero solution). This follows readily by noting that if  $\{a_i\}$  are solutions to the generating equations (Eqs. 9) for some  $\{h_{ij}\}$ , then  $\{\frac{1}{k} a_i\}$  are solutions for  $\{kh_{ij}\}$ . The interpretation of this is quite simple. The IRT

$$s(t) = \sum_{n=N_1}^{N_2} a_n h(t-(n-1)T)$$

could just as easily be generated with the kernel scaled by  $k$  and the  $\{a_n\}$  scaled by  $1/k$ . That is,

$$s(t) = \sum_{n=N_1}^{N_2} \frac{a_n}{k} kh(t-(n-1)T).$$

Thus, whenever the homogeneous equations have a 1-parameter infinity of solutions, and one of these solutions is acceptable, they must all be acceptable (with the single noted exception of the zero solution). So it is permissible to speak of a 1-parameter acceptable solution.

If it is somehow known in advance that a set of homogeneous equations will have a 1-parameter solution, then one of the unknowns can be assigned a value before solving the equations, for example,  $h_{1,1} = 1$ . [It is assumed the value of the unknown is not constrained to be zero by the equations (see Sec. VI).] The resulting equations will no longer be homogeneous, and will have a unique solution for the remaining unknowns. Thus the unknown to which the value was assigned becomes a norm for all the rest of the unknowns, scaling the whole solution to its assigned value. This situation is especially desirable since there is no doubt about which of the infinity of solutions is to be selected — one is as good as the other, since they are just scaled versions of each other.

Besides the properties already noted, 1-parameter solutions to the homogeneous equations (acceptable or not) have another interesting and useful property. This property follows from a theorem that we shall prove (Theorem 3), which asserts that if the homogeneous equations for some span  $D$  have a 1-parameter solution, then

- (i) the homogeneous equations for spans  $> D$  have a  $> 1$ -parameter solution,
- (ii) the homogeneous equations for spans  $< D$  have only the trivial solution.

What this means is that if for some span the homogeneous equations have a 1-parameter solution, then it is the only span for which they have a 1-parameter solution. Furthermore, it is the minimum span for which a nontrivial solution exists. Therefore, if a 1-parameter acceptable solution is known to exist for some span, it must be the acceptable solution of minimum span. We shall prove that for waveforms whose beginning and ending are known there is always guaranteed to be a 1-parameter acceptable solution for some span — and we are now guaranteed by Theorem 3 that it is the acceptable solution of minimum span.

**THEOREM 3.** If a set of homogeneous equations has a 1-parameter solution, (i) the homogeneous equations for spans  $> D$  have a  $> 1$ -parameter solution, (ii) the homogeneous equations for spans  $< D$  have only the trivial solution.

Proof: (i) Suppose that the coefficient matrix of the homogeneous equations for span  $D$  has rank  $2D - 1$ , that is,  $\text{rank} \begin{bmatrix} S_D^N \end{bmatrix} = 2D - 1$ . Then in the coefficient matrix for span  $D + 1$ , the first  $2D$  columns are columns of  $\begin{bmatrix} S_D^N \end{bmatrix}$  without the last row, and the last  $2D$  columns are the columns of  $\begin{bmatrix} S_D^N \end{bmatrix}$  without the first row. Therefore, the first and last columns of  $\begin{bmatrix} S_{D+1}^N \end{bmatrix}$  are dependent upon the center  $2D$  columns. Therefore  $\text{rank} \begin{bmatrix} S_{D+1}^N \end{bmatrix} \leq 2D$ , and the homogeneous equations have a 2-parameter or greater solution. /

(ii) Now suppose that  $\text{rank} \begin{bmatrix} S_D^N \end{bmatrix} = 2D - 1$ , that is, the homogeneous equations  $\begin{bmatrix} S_D^N \\ H_D \end{bmatrix} = 0$ , have a 1-parameter solution. If there is a set of homogeneous equations

for the same sample pairs with lesser span which has a P-parameter solution,  $P \geq 1$ , then, by the previous result, the equations for span D cannot have a 1-parameter solution. Since this contradicts the assumption that  $\text{rank} \begin{bmatrix} S_D^N \end{bmatrix} = 2D - 1$ , no homogeneous equations for span  $< D$  can have nontrivial solutions. Therefore, if a set of homogeneous equations for some span D have a 1-parameter solution, then it is the nontrivial solution of minimum span. //

### 3.4 SOLUTIONS FOR WAVEFORMS WHOSE BEGINNING AND ENDING ARE KNOWN

We now demonstrate that for waveforms whose beginning and ending are known, that is, waveforms of finite duration, there always exists a 1-parameter acceptable solution to the homogeneous equations written for some span.

The phrase, "whose beginning and ending are known," requires better definition. Let  $s(t)$  be a waveform known on a finite interval divided into N periods, and let  $s(t)$  be zero for D - 1 periods at the beginning and ending of the interval. Then  $s(t)$  is called a complete waveform. Note that completeness depends upon the value of D (see Fig. 9). Thus we shall refer to D-complete waveforms. If an IRT  $s(t)$  with kernel  $h(t)$  of span D

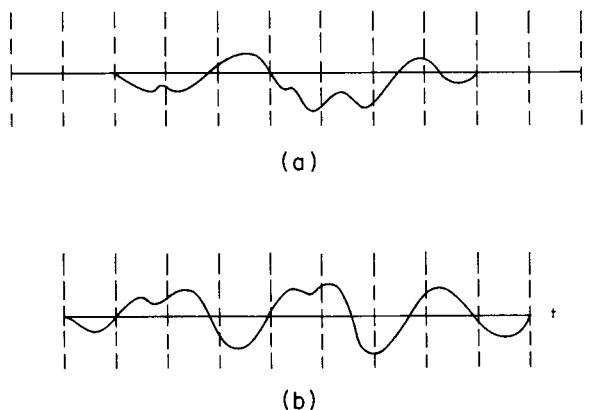


Fig. 9. Complete waveforms: (a) for  $D = 3$ ;  
(b) for  $D = 1$ .

is D-complete, it is clear that no impulse response starting outside the beginning of the interval can make a nonzero contribution to  $s(t)$ , hence the "beginning" of  $s(t)$  is known. Similarly, the condition also implies that the last impulse response occurring within the interval cannot make any nonzero contribution outside the interval, hence the "ending" of  $s(t)$  is known. When a waveform is complete, and the periods are numbered from 1 to N, the homogeneous equations take on a particular form. In the matrix  $\begin{bmatrix} S_D^N \end{bmatrix}$ , the first row has zeros in its first  $2(D-1)$  entries, and both of the remaining two entries cannot be zero; the second row has zeros in its first  $2(D-2)$  entries, and not all of the



remaining entries can be zero; etc. At the bottom of the matrix, the last row has zeros in its last  $2(D-1)$  entries, and not all of the remaining entries can be zero; the second from the last row has zeros in its last  $2(D-1)$  entries, and not all of the remaining entries can be zero; etc.

For example, if we assume that the period sample pairs for  $N = 9$  and  $D = 3$  are

$$\begin{array}{ll} s_{1,1} = 0 & s_{1,2} = 0 \\ s_{2,1} = 0 & s_{2,2} = 0 \\ s_{3,1} = 1 & s_{3,2} = 2 \\ s_{4,1} = 3 & s_{4,2} = 3 \\ s_{5,1} = 1 & s_{5,2} = -3 \\ s_{6,1} = -1 & s_{6,2} = 3 \\ s_{7,1} = 0 & s_{7,2} = -1 \\ s_{8,1} = 0 & s_{8,2} = 0 \\ s_{9,1} = 0 & s_{9,2} = 0 \end{array}$$

then the coefficient matrix  $\begin{bmatrix} S_3^9 \end{bmatrix}$  of the corresponding homogeneous equations is

$$\begin{bmatrix} S_3^9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & 3 & -3 \\ 1 & -2 & 3 & -3 & 1 & 3 \\ 3 & -3 & 1 & 3 & -1 & -3 \\ 1 & 3 & -1 & -3 & 0 & 1 \\ -1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, since for a  $D$ -complete IRT the "beginning" and "end" of the IRT are known, it might be expected for a kernel of span  $D$  that the impulse areas before the  $D^{\text{th}}$  period and after the  $(N-2D+2)^{\text{th}}$  period would be zero. If this were not true, the IRT would be expected to be nonzero outside the interval between its assumed "beginning" and "ending."

The following is a corollary of Theorem 2 to show that these impulse areas are zero for solutions satisfying the condition  $|H_D| \neq 0$ .

**COROLLARY OF THEOREM 2.** Let  $\{s_{1,1}, s_{1,2}, \dots, s_{N,1}, s_{N,2}\}$  be any set of  $N$  consecutive sample pairs from a complete waveform  $s(t)$ . Let  $\{h_{D,2}, h_{D,1}, \dots, h_{1,2}, h_{1,1}\}$  be a solution set of the homogeneous equations

$\begin{bmatrix} S_D^N \\ H_D \end{bmatrix} = 0$  such that  $|H_D| \neq 0$ . Then there exists a unique solution for the generating equations  $S^N = \begin{bmatrix} H_D^N \\ A_D^N \end{bmatrix}$  such that  $a_i = 0$ ,  $-D + 2 \leq i < D$  and  $N - 2D + 2 < i \leq N$ .

Proof: There exists a unique solution for the generating equations, by Theorem 2. /

The assertion that  $a_i = 0$ ,  $-D + 2 \leq i < D$ , and  $N - 2D + 2 < i \leq N$  can be proved as follows. Consider the  $2(D-1)$  generating equations for the first  $D - 1$  periods. Since  $s(t)$  is complete, these equations are homogeneous and the determinant of their coefficient matrix is  $|H_D|$ , which is nonzero by assumption. Hence the solutions are zero only. Therefore  $a_i = 0$ ,  $-D + 2 \leq i < D$ .

Now, consider the  $2(D-1)$  generating equations for the last  $2(D-1)$  periods. Again they are homogeneous and the determinant of their coefficient matrix is  $|H_D| \neq 0$ . Hence the solutions are zero only. Therefore  $a_i = 0$ ,  $N - 2D + 2 < i \leq N$ . //

The theorem that will be proved next will show that for a complete waveform the acceptable solution of minimum span is 1-parameter and gives unique impulse areas.

**THEOREM 4.** Let the homogeneous equations  $\begin{bmatrix} S_D^N \\ H_D \end{bmatrix} = 0$ ,  $D \geq 2$ , for the periodic sample pairs of a complete waveform, periods numbered 1 to  $N$ , have rank  $< 2D$ . Then (i) If the rank of the equations is  $< 2D - 1$ , the homogeneous equations for some span  $< D$  have a 1-parameter solution. (ii) If the rank of the equations is  $2D - 1$  (that is, they have a 1-parameter solution), their solution is acceptable. Furthermore, the impulse areas corresponding to this solution (for a fixed norm) are unique with  $a_i = 0$ ,  $-D + 2 \leq i < D$  and  $N - 2D + 2 < i \leq N$ .

Proof: The proof will be by induction, that is, the theorem will be proved for  $D = 2$  and then shown to hold for any  $D$  if it holds for  $D - 1$ .

Assume  $D = 2$ . Let  $\{h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}\}$  be one of the nontrivial solutions to the homogeneous equations  $\begin{bmatrix} S_2^N \\ H_2 \end{bmatrix} = 0$ . These equations are

$$\begin{aligned} s_{2,1}h_{1,2} - s_{2,2}h_{1,1} &= 0 \\ s_{2,1}h_{2,2} - s_{2,2}h_{2,1} + s_{3,1}h_{1,2} - s_{3,2}h_{1,1} &= 0 \\ \vdots & \\ s_{N-2,1}h_{2,2} - s_{N-2,2}h_{2,1} + s_{N-1,1}h_{1,2} - s_{N-1,2}h_{1,1} &= 0 \\ s_{N-1,1}h_{2,2} - s_{N-1,2}h_{2,1} &= 0 \end{aligned} \tag{14}$$

(i) Suppose that the general solution to the homogeneous equations is more than 1-parameter, that is, rank  $\begin{bmatrix} S_2^N \\ H_2 \end{bmatrix} < 3$ . It will now be shown that rank  $\begin{bmatrix} S_1^N \\ H_1 \end{bmatrix} < 2$ . For, suppose that rank  $\begin{bmatrix} S_1^N \\ H_1 \end{bmatrix} = 2$ . Since the columns of  $\begin{bmatrix} S_1^N \\ H_1 \end{bmatrix}$  are the first two columns of  $\begin{bmatrix} S_2^N \\ H_2 \end{bmatrix}$ , the first two columns of  $\begin{bmatrix} S_2^N \\ H_2 \end{bmatrix}$  are linearly independent. Then if  $s_{2,1} \neq 0$ , columns 1, 2,

and 3 of  $\begin{bmatrix} S_2^N \end{bmatrix}$  must be linearly independent, since there are only zeros in the first row of columns 1 and 2. Hence  $\text{rank} \begin{bmatrix} S_2^N \end{bmatrix} = 3$ , which is a contradiction. Therefore  $\text{rank} \begin{bmatrix} S_1^N \end{bmatrix} < 2$ .

If  $s_{2,1} = 0$ , then  $s_{2,2} \neq 0$ , by assumption. Then columns 1, 2, and 4 of  $\begin{bmatrix} S_2^N \end{bmatrix}$  are independent, and the same result follows.

But  $\text{rank} \begin{bmatrix} S_1^N \end{bmatrix} \neq 0$ , since it is assumed that the entries of  $\begin{bmatrix} S_1^N \end{bmatrix}$  are not all zero.

Therefore  $\text{rank} \begin{bmatrix} S_1^N \end{bmatrix} = 1$ , and the equations  $\begin{bmatrix} S_1^N \end{bmatrix} H_1 = 0$  have a 1-parameter solution. /

(ii) If the general solution to the homogeneous equations is 1-parameter, then it can be shown that  $|H_2| \neq 0$ .

For, suppose that  $|H_2| = 0$ . Then

$$\begin{vmatrix} h_{2,1} & h_{1,1} \\ h_{2,2} & h_{1,2} \end{vmatrix} = 0.$$

It is convenient to interpret this as meaning that the two columns are multiples of some other column  $\begin{bmatrix} g_{1,1} \\ g_{1,2} \end{bmatrix}$ , where neither of  $g_{1,1}, g_{1,2}$  are zero. Thus

$$\begin{aligned} h_{1,1} &= b_1 g_{1,1} & h_{2,1} &= b_2 g_{2,1} \\ h_{1,2} &= b_1 g_{1,2} & h_{2,2} &= b_2 g_{2,2} \end{aligned}.$$

Substituting these values in the homogeneous equations (Eqs. 14) gives

$$\begin{aligned} b_1 (s_{2,1} g_{1,2} - s_{2,2} g_{1,1}) &= 0 \\ b_2 (s_{2,1} g_{1,2} - s_{2,2} g_{1,1}) + b_1 (s_{3,1} g_{1,2} - s_{3,2} g_{1,1}) &= 0 \\ \vdots & \\ b_2 (s_{N-2,1} g_{1,2} - s_{N-2,2} g_{1,1}) + b_1 (s_{N-1,1} g_{1,2} - s_{N-1,2} g_{1,1}) &= 0 \\ b_2 (s_{N-1,1} g_{1,2} - s_{N-1,2} g_{1,1}) &= 0. \end{aligned}$$

Since both of  $b_1$  and  $b_2$  cannot be zero (otherwise the  $h_{i,j}$  would be zero, but it has been assumed that the solution is nontrivial), these equations reduce to the homogeneous equations

$$\begin{aligned} s_{2,1} g_{1,2} - s_{2,2} g_{1,1} &= 0 \\ s_{3,1} g_{1,2} - s_{3,2} g_{1,1} &= 0 \\ \vdots & \\ s_{N-1,1} g_{1,2} - s_{N-1,2} g_{1,1} &= 0. \end{aligned}$$

Therefore, in the matrix  $\begin{bmatrix} S_2^N \end{bmatrix}$  columns 1 and 2 are dependent, and columns 3 and 4 are dependent. Therefore the maximum number of independent columns is 2 and of rank  $\begin{bmatrix} S_2^N \end{bmatrix} < 3$ , which contradicts the assumption that  $\text{rank} \begin{bmatrix} S_2^N \end{bmatrix} = 3$ . Therefore  $|H_2| \neq 0$ .

Then, by the corollary of Theorem 2 there exists a unique solution for the generating equations  $\begin{bmatrix} S_2^N \end{bmatrix} = \begin{bmatrix} H_2^N \end{bmatrix} A_2^N$  such that  $a_i = 0$ ,  $-D + 2 \leq i < D$  and  $N - 2D + 2 < i \leq N$ . /

Now the theorem will be proved for any span  $D$ , under the assumption that it holds for all spans from 2 to  $D - 1$ .

(i) Assume that the general solution to the homogeneous equations is more than 1-parameter. Then  $\text{rank} \begin{bmatrix} S_D^N \end{bmatrix} < 2D - 1$ . It can now be shown that the first (or the last)  $2(D-1)$  columns of  $\begin{bmatrix} S_D^N \end{bmatrix}$ , which are the coefficients of the homogeneous equations for a span of  $D - 1$ , cannot have rank  $2(D-1)$ .

For, suppose that the rank of the first  $2(D-1)$  columns is  $2(D-1)$ . Then, if  $s_{D,1} \neq 0$ , columns 1 through  $2D - 1$  must be linearly independent, since column  $2D - 1$  has a non-zero entry,  $s_{D,1}$ , in a row where the first  $2(D-1)$  columns have zero entries. If  $s_{D,1} = 0$ , then  $s_{D,2} \neq 0$  by assumption, and hence, by the same argument, the first  $2(D-1)$  columns and the last column are linearly independent. But, in either case, this means that  $\text{rank} \begin{bmatrix} S_D^N \end{bmatrix} = 2D - 1$ , which is false by assumption. Therefore, the homogeneous equations for span  $D - 1$  are of rank  $< 2(D-1)$ .

Now, the homogeneous equations for some span less than  $D$  must have a 1-parameter solution. Otherwise, since the theorem applies for every span less than  $D$ , application at each successive level eventually implies  $\text{rank} \begin{bmatrix} S_1^N \end{bmatrix} = 0$ , which is impossible because the entries of  $\begin{bmatrix} S_1^N \end{bmatrix}$  are not all zero. /

(ii) Assume that the homogeneous equations  $\begin{bmatrix} S_D^N \end{bmatrix} H_D = 0$  have a 1-parameter solution. Then the determinant  $|H_D| \neq 0$ .

For, suppose that  $|H_D| = 0$ . Then consider the corresponding matrix  $\begin{bmatrix} H_D^{D-1} \end{bmatrix}$ . By performing the three following operations on this matrix, it can be converted to a matrix of coefficients of a set of homogeneous equations of order  $2(D-1)$  for periodic sample pairs of a complete waveform for span  $D - 1$ .

(a) Take the transpose.

(b) Rotate the matrix about a horizontal center line, that is, interchange row  $m$  and row  $2(D-1) - m$ ;  $1 \leq m \leq D - 1$ .

(c) Multiply the even numbered columns by  $-1$ .

Since these operations can do no more than change the sign of the determinant, the determinant is still zero. Hence the corresponding homogeneous equations have a non-trivial solution. Now, since the theorem is true for all spans up to  $D - 1$ , there exists some span  $X < D$  for which the homogeneous equations for the periodic sample pairs that

are the entries of  $H_D]$  have 1-parameter acceptable solution with unique impulse areas. Thus the elements of the matrix  $H_D]$  can be written as samples of a complete IRT with kernel  $g(t)$  of span  $X < D$ . The matrix  $H_D]$  can therefore be expanded in terms of the matrix  $G_X]$  and the impulse areas  $\{b_i\}_{i=1}^{D-X+1}$  as follows.

$$\begin{bmatrix} h_{D,2} \\ h_{D,1} \\ . \\ . \\ . \\ h_{1,2} \\ h_{1,1} \end{bmatrix} = b_1 \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ G_X \\ . \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ . \\ . \\ . \\ G_X \\ . \\ 0 \end{bmatrix} + \dots + b_{D-X+1} \begin{bmatrix} G_X \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

Substituting this expansion in the homogeneous equations  $\begin{bmatrix} S_D^N \\ G_X \end{bmatrix} = 0]$  gives

$$b_{D-X+1} \begin{bmatrix} 0 \\ . \\ . \\ . \\ . \\ \begin{bmatrix} S_X^N \\ G_X \end{bmatrix} \\ 0 \end{bmatrix} + b_{D-X} \begin{bmatrix} 0 \\ . \\ . \\ . \\ \begin{bmatrix} S_X^N \\ G_X \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} + \dots + b_1 \begin{bmatrix} \begin{bmatrix} S_X^N \\ G_X \end{bmatrix} \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} = 0]$$

Now it can be shown that this expression implies that  $\begin{bmatrix} S_X^N \\ G_X \end{bmatrix} = 0]$ .

Assume that  $b_1 \neq 0$ . Then consider the first entry of the last matrix, which is the first entry of  $\begin{bmatrix} S_X^N \\ G_X \end{bmatrix}$ . Clearly, it must be zero, since the first entry in all of the other terms is zero. Consider the second entry of the last matrix. This entry plus the second entry of the second from the last matrix is zero. But the second entry of the second from the last matrix is the same as the first entry of the last matrix, which has just been shown to be zero. Hence the second entry of the last matrix is zero. This process can be continued until all of the entries of the last matrix are shown to be zero. Hence  $\begin{bmatrix} S_X^N \\ G_X \end{bmatrix} = 0]$ .

If  $b_1 = 0$ , the same process can be applied to the first term on the right for which

$b_i \neq 0$ , with the same result.

Since  $|G_X| \neq 0$ , the equation  $\begin{bmatrix} S_X^N \\ G_X \end{bmatrix} = 0$  implies that the columns of  $\begin{bmatrix} S_X^N \end{bmatrix}$  are linearly dependent. This implies that in the matrix  $\begin{bmatrix} S_D^N \end{bmatrix}$ , the first column is dependent upon the  $2X - 1$  columns which follow it, and the last column is dependent upon the  $2X - 1$  columns which precede it. Hence the rank of  $\begin{bmatrix} S_D^N \end{bmatrix}$  cannot exceed  $2(D-1)$ . But this is a contradiction since it was assumed that  $\text{rank} \begin{bmatrix} S_D^N \end{bmatrix} = 2D - 1$ . Hence  $|H_D| \neq 0$ .

Then, by the corollary of Theorem 2 there exists a unique solution for the generating equations  $S_D^N = \begin{bmatrix} H_D^N \\ A_D^N \end{bmatrix}$  such that  $a_i = 0$ ,  $-D + 2 \leq i < D$  and  $N - 2D + 2 < i \leq N$ . //

This theorem shows that for a complete waveform there is always a 1-parameter acceptable solution. For, by Theorem 1 there is always an acceptable solution, and by this theorem either that acceptable solution is 1-parameter, or there is a 1-parameter solution for some lesser span. Furthermore, by Theorem 3, this 1-parameter solution is the acceptable solution of minimum span.

### 3.5 AN APPROACH TO FINDING THE SPAN OF THE MINIMUM KERNEL

The results that we have achieved thus far suggest a way of finding the span of the minimum kernel for IRT of finite duration. Suppose that the homogeneous equations are set up and solved for spans  $D = 1, 2, 3, \dots$ , etc. Then, by Theorem 3, as long as the span is less than the span for which there is a 1-parameter solution, the equations will have only the trivial solution. As the span is increased, however, there must eventually be a nontrivial solution, by Theorem 1, and this must be the 1-parameter solution, and by Theorem 4 it must be acceptable. The approach, then, is to look for the span for which the first nontrivial solution occurs. But we should like to propose a variation of this approach which is closer to the method that we shall actually use because it has advantages for machine computation.

It is possible to make another interesting observation from Theorem 4. One of the key points in proving the theorem was showing that any solution to the homogeneous equations written for a span greater than the minimum could always be represented as the samples of an IRT with kernel of shorter span. This process continued until it was ultimately shown that the acceptable solution of minimum span consists in samples of a kernel that can generate any solution for a higher span. This means that the minimum kernel will always be able to generate any kernel of larger span as an IRT.

Since we are going to look for a 1-parameter acceptable solution, we might as well choose one of the unknowns as the norm (we shall choose  $h_{1,1}$ ) and constrain it to have a nonzero value. Now we set up and solve the homogeneous equations for span  $D = 1, 2, 3, \dots$ , etc. Since the equations for spans less than the span for which there is a 1-parameter solution can have only the trivial solution, the assigning of a nonzero

value to  $h_{1,1}$  will cause these equations to have no solution for this constraint. On the other hand, when the span for which there is a 1-parameter solution is reached, the equations will be solved and give a unique solution. Thus, when one of the unknowns is constrained to be nonzero, the lowest span for which a solution exists is also the lowest span for which an acceptable solution exists.

It is tempting to say at this point that the lowest span for which the 1-parameter acceptable solution exists is the span of the minimum kernel. Unfortunately, we are not justified in saying this because it may be that this span is lower than the span of the minimum kernel. Suppose, for example, that the homogeneous equations were written for an IRT whose minimum kernel had span 2. Furthermore, suppose that the coincident samples of the second period of the kernel happened both to be zero for the chosen set of periodic sample pairs of the IRT. Then it is clear that the homogeneous equations written for span 1 would have a solution, namely, the coincident samples in the first period of the kernel. Thus, it is not possible to determine the span of the minimum kernel by solving just this set of homogeneous equations. For span 2, however, the homogeneous equations written for the vast majority of periodic sample pairs will have 1-parameter solutions, so the span of the minimum kernel can be found by solving many sets of homogeneous equations written for different choices of periodic sample pairs.

#### IV. SOLVING THE HOMOGENEOUS EQUATIONS IN THE PRESENCE OF NOISE

Since it is our ultimate goal to apply the theory of IRT to physically generated signals, we must be prepared to deal with perturbations of these physical signals from the IRT idealization. These perturbations could result from any number of unknown factors, such as additive noise, variations in the pulse spacing, or time variations in the impulse response. The net effect of these perturbations from the ideal will be to cause the homogeneous equations to have only the trivial solution (except for the uninteresting case in which we let the impulse response be the signal itself), which is of no use to us.

To avoid this, we shall take the point of view that the signal that we are attempting to analyze really is an IRT, but it has been altered "slightly," because of various disturbances, which will be represented by additive noise. Then we should like to form an estimate of the kernel of this underlying IRT. One approach to securing this estimate is to note that if the departure from the ideal is "small," the homogeneous equations written for the span of the kernel of the underlying IRT should "almost" have a solution, that is, there is some set of numbers  $\{h_{ij}\}$  that make the left-hand side of the equations "nearly" zero. We propose, then, as a reasonable approach, not to try to solve the homogeneous equations outright, but rather to find that set of numbers which comes "closest" to solving the equations, in the sense that it minimizes the mean-square value of the left-hand side of the equations. For example, for  $D = 2$  this means finding the  $\{h_{ij}\}$  that minimizes the mean square of the set of residuals  $r_p$ , where

$$\{s_{p,1}h_{2,2} - s_{p,1}h_{2,1} + s_{p+1,1}h_{1,2} - s_{p+1,2}h_{1,1} = r_p\}.$$

This leads to difficulties immediately because the obvious solution is to make all of the  $h_{ij} = 0$ , in which case  $\{r_p = 0\}$ ! In order to avoid this undesirable result, let us suppose that the underlying IRT, if it were not perturbed from the ideal, is of the type that has a 1-parameter acceptable solution for some span. Then, proceeding as we did previously, we assign a nonzero value to one of the  $h_{ij}$ , and this coincident sample of the kernel becomes a norm that scales all of the other samples. This solves our problem, if, for example, we were to set  $h_{1,1} = 1$  in the example for  $D = 2$ , then setting the other  $h_{ij} = 0$  no longer minimizes the mean square of the  $r_p$ , but some other values must be sought.

One effect of the perturbations has been to rob us of our former technique (section 3.5) for determining the span of the 1-parameter solution. It is no longer of any value to check for the minimum span for which a solution exists when one of the  $h_{ij}$  is constrained to be nonzero, since the only exact solutions are now the trivial solutions. A similar procedure is still available, however, and will be discussed below.

Now we turn to the details of the method for finding the "best" solution to the homogeneous equations.



#### 4.1 ESTIMATING THE SOLUTIONS TO THE HOMOGENEOUS EQUATIONS

We shall assume that a zero-mean noise signal  $e(t)$  is added to an IRT  $s(t)$ . The addition of this noise gives rise to the perturbed IRT  $\tilde{s}(t) = s(t) + e(t)$ , from which we desire to find a best estimate  $\hat{h}(t)$  of the true minimum kernel,  $h(t)$ . It is therefore necessary to define the meaning of best estimate.

The first step is to discover the effect of the noise on the homogeneous equations. For convenience, the simplest case, for  $D = 1$ , will be used in the discussion. For  $N$  periods of  $s(t)$  the set of homogeneous equations for  $D = 1$  is

$$\left\{ s_{i,1} h_{1,2} - s_{i,2} h_{1,1} = 0 \right\}_{i=1}^N. \quad (15)$$

The noise signal  $e(t)$  causes the observed samples to deviate from the true values. The true values can be written in terms of the perturbed samples as follows.

$$\left\{ \begin{array}{l} s_{i,1} = \tilde{s}_{i,1} - e_{i,1} \\ s_{i,2} = \tilde{s}_{i,2} - e_{i,2} \end{array} \right\}_{i=1}^N \quad (16)$$

Substituting these in Eqs. 13 gives

$$\left\{ (\tilde{s}_{i,1} - e_{i,1}) h_{1,2} - (\tilde{s}_{i,2} - e_{i,2}) h_{1,1} = 0 \right\}_{i=1}^N$$

or

$$\left\{ \tilde{s}_{i,1} h_{1,2} - \tilde{s}_{i,2} h_{1,1} = h_{1,2} e_{i,1} - h_{1,1} e_{i,2} \right\}_{i=1}^N$$

or

$$\left\{ \tilde{s}_{i,1} h_{1,2} - \tilde{s}_{i,2} h_{1,1} = r_i \right\}_{i=1}^N, \quad (17)$$

where  $r_i = h_{1,2} e_{i,1} - h_{1,1} e_{i,2}$ .

Since the residuals  $r_i$  are unknown, Eqs. 15 cannot be solved even if they are consistent, which they almost certainly will not be for  $N > 2$ . Hence the true values of the quantities  $h_{1,1}$  and  $h_{1,2}$  must remain unknown.

Primarily as a matter of convenience, the criterion used to define the best estimate of the true values of the unknowns  $h_{1,1}$  and  $h_{1,2}$  shall be that the estimated values,  $\hat{h}_{1,1}$  and  $\hat{h}_{1,2}$ , minimize the mean-square values of the residuals  $r_i$ .<sup>7</sup>

The quantity to be minimized then is

$$\frac{1}{N} \sum_{i=1}^N r_i^2 = \frac{1}{N} \sum_{i=1}^N (\tilde{s}_{i,1} h_{1,2} - \tilde{s}_{i,2} h_{1,1})^2$$

or, with the usual notation for the average,

$$\overline{r_i^2} = \overline{(\tilde{s}_{i,1} h_{1,2} - \tilde{s}_{i,2} h_{1,1})^2}. \quad (18)$$

Setting the partial derivatives of this expression with respect to  $h_{1,1}$  and  $h_{1,2}$  to zero for a minimum gives

$$\begin{aligned} \overline{\tilde{s}_{i,1}^2} h_{1,2} - \overline{\tilde{s}_{i,1} \tilde{s}_{i,2}} h_{1,1} &= 0 \\ \overline{\tilde{s}_{i,2} \tilde{s}_{i,1}} h_{1,2} - \overline{\tilde{s}_{i,2}^2} h_{1,1} &= 0. \end{aligned} \quad (19)$$

Since, in general, there is no reason for these equations to be dependent, the solution to these equations as they stand is  $\hat{h}_{1,2} = \hat{h}_{1,1} = 0$ , which minimizes  $\overline{r_i^2}$ , but is not very satisfying. To avoid this result, the value of one of the unknowns will be fixed and used as a norm.

If the variable  $h_{1,2}$  is chosen as the unity norm, then there is no derivative with respect to this variable, and the first of Eqs. 19 does not exist, which leaves

$$\overline{\tilde{s}_{i,2} \tilde{s}_{i,1}} - \overline{\tilde{s}_{i,2}^2} h_{1,1} = 0 \quad (20)$$

to be solved for the best estimate of  $h_{1,1}$ . This optimum value is

$$\hat{h}_{1,1} = \frac{\overline{\tilde{s}_{i,2} \tilde{s}_{i,1}}}{\overline{\tilde{s}_{i,2}^2}}.$$

If the variable  $h_{1,1}$  is chosen as the unity norm, Eqs. 19 reduce to

$$\overline{\tilde{s}_{i,1}^2} h_{1,2} - \overline{\tilde{s}_{i,1} \tilde{s}_{i,2}} = 0 \quad (21)$$

which has the solution

$$\hat{h}_{1,2} = \frac{\overline{\tilde{s}_{i,1} \tilde{s}_{i,2}}}{\overline{\tilde{s}_{i,1}^2}},$$

which is not necessarily the same as the solution to Eqs. 20. In general, the solution will depend upon which variable is chosen as the norm. This irritation can be eliminated without much inconvenience in the simple case of  $D = 1$  by adding the constraint that the estimated values of the noise added to both samples of a given sample pair be equal. In that case the solution is independent of whichever variable is chosen as the

norm. For  $D \geq 2$ , however, the cases of most interest, this constraint yields non-linear equations that are not at all convenient. Therefore the dependency of the estimated values upon the choice of the norm must be accepted if the convenience of linear equations is to be retained.

Anyone familiar with statistics will recognize that the mechanics of finding  $\hat{h}_{1,1}$  and  $\hat{h}_{1,2}$  is the same as that for finding the linear regression line for the set of points  $\{\tilde{s}_{i,1}, \tilde{s}_{i,2}\}_{i=1}^N$  which is constrained to pass through the origin. Equations 20 and 21 would correspond to different assumptions about the direction in which the mean-square error is to be minimized; that is, whether the error is in the first or second sample of the sample pairs. The assumptions that form the starting point for the derivation of Eqs. 20 and 21 are quite different from those from which the regression line is derived. The following graphical interpretation is intended to give some insight into the derivation of Eqs. 20 and 21.

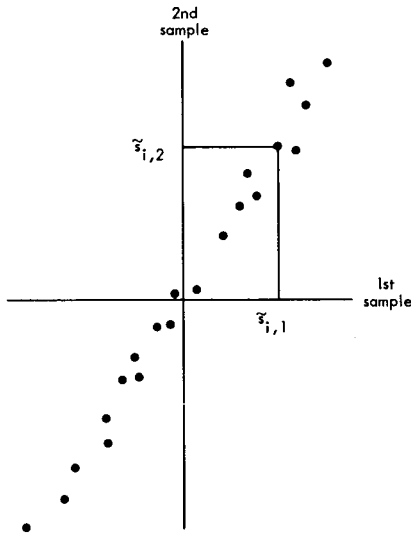


Fig. 10. Periodic sample pair space for span 1.

Figure 10 is a plane on which points whose coordinates are the periodic sample pairs  $\{s_{i,1}, s_{i,2}\}_{i=1}^N$  are plotted. If there were no noise, and these really were the samples of an IRT having a kernel of span  $D = 1$ , then these points must all lie on a straight line through the origin. This is so because the homogeneous equations are the equations of a straight line with zero intercept. The slope of this line is the ratio of the kernel sample pairs which are coincident with the sample pairs of the IRT. Thus, finding this straight line is equivalent to finding the desired samples of the kernel.

In Fig. 10 we assume that the coordinates of the points are periodic sample pairs of an IRT with kernel of span  $D = 1$ , with noise added to it. Because of the added noise, these points do not lie on a straight line through the origin. Then the problem is to fit a best line to the set of points. A graphical interpretation of the process by which this

"best" line is determined is worth studying.

It has been decided that the "best" line is to be determined by choosing the values of  $\hat{h}_{1,1}$  and  $\hat{h}_{1,2}$  that minimize the mean of the squares of the residuals,  $\overline{r_i^2}$ . Let the values of the residuals  $r_i$  corresponding to the values  $\hat{h}_{1,1}$  and  $\hat{h}_{1,2}$  of the unknowns be  $\hat{r}_i$ . Now, corresponding to the sample pair point  $(\tilde{s}_{i,1}, \tilde{s}_{i,2})$  we can visualize an "estimated sample pair point"  $(\hat{s}_{i,1}, \hat{s}_{i,2})$  lying on the "best" line:

$$\hat{s}_{i,1}\hat{h}_{1,2} - \hat{s}_{i,2}\hat{h}_{1,1} = 0. \quad (22)$$

This point is the best estimate of the true value of the sample point before the noise was added. The exact location of this point cannot be determined, however, since it depends upon the choice of impulse areas. Then, the estimated values of the noise,  $\hat{e}_{i,1}$  and  $\hat{e}_{i,2}$ , are the differences between the estimated and measured sample pair points:

$$\begin{aligned} \hat{e}_{i,1} &= \tilde{s}_{i,1} - \hat{s}_{i,1} \\ \hat{e}_{i,2} &= \tilde{s}_{i,2} - \hat{s}_{i,2} \end{aligned} \quad (23)$$

The values of these estimated errors cannot be determined either, until the impulse areas are chosen. Figure 11 shows possible locations for the measured and estimated

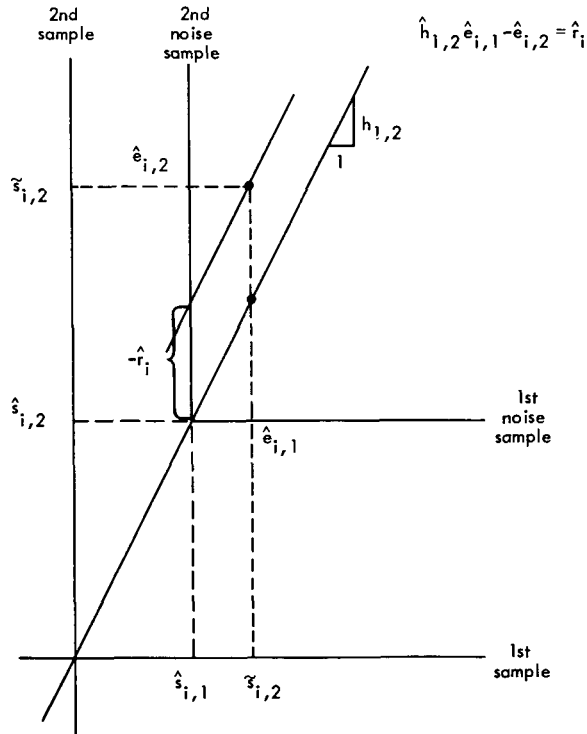


Fig. 11. Graphical interpretation of the residual  $\hat{r}_i$  with  $h_{1,1}$  as norm.

points in the sample pair plane, along with the "best" line given by Eq. 22. Equations 23 show that an "estimated noise sample pair" plane can be superimposed upon the sample

pair plane, with a set of coordinates having their origin at the point  $(\hat{s}_{i,1}, \hat{s}_{i,2})$ . The point  $(\tilde{s}_{i,1}, \tilde{s}_{i,2})$  in the sample pair plane is the point  $(e_{i,1}, e_{i,2})$  in the estimated noise sample pair plane. The residual  $\hat{r}_i$  is given by

$$\hat{r}_i = \hat{h}_{i,2} \hat{e}_{i,1} - h_{i,1} \hat{e}_{i,2} \quad (24)$$

which is a straight line in the estimated noise sample plane. If  $h_{1,1}$  is designated as the unity norm, the equation of this line becomes

$$\hat{s}_{i,2} = \hat{h}_{1,2} \hat{e}_{i,1} - \hat{r}_i, \quad (25)$$

which is plotted in Fig. 12. This diagram makes it clear that the "best" line is the one that minimizes the mean square of the vertical distance from the line to the sample pair

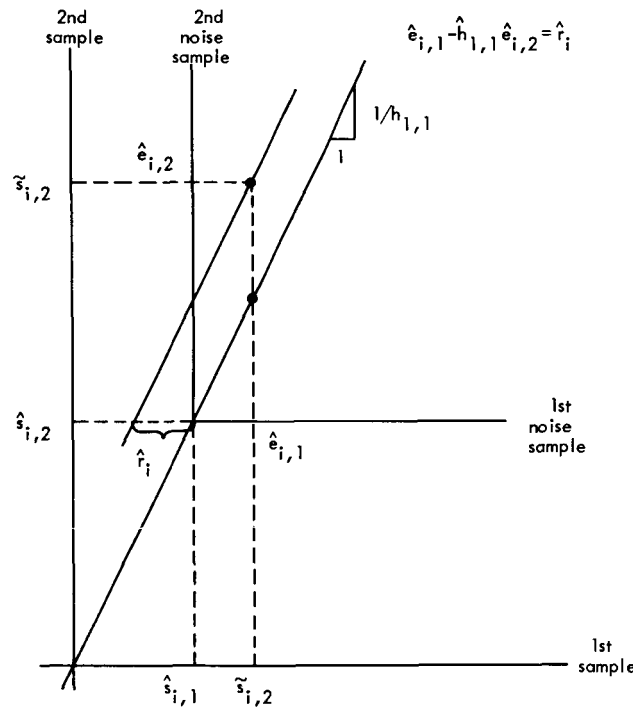


Fig. 12. Graphical interpretation of the residual  $\hat{r}_i$  with  $h_{1,2}$  as norm.

point. Similarly, if  $h_{1,2}$  is chosen as unity norm, Eq. 24 becomes

$$\hat{e}_{i,1} = \hat{h}_{1,1} \hat{e}_{i,2} - \hat{r}_i,$$

which is plotted in Fig. 13. In this case, the best line is the one that minimizes the mean square of the horizontal distance from the line to the sample pair point. From the geometrical interpretation, it is obvious that the "best" line will depend upon the choice of norm.

From the preceding discussion and Figs. 12 and 13, it is clear that the location of the estimated sample pair point  $(\hat{s}_{i,1}, \hat{s}_{i,2})$  cannot be determined from the knowledge of

$(\tilde{s}_{i,1}, \tilde{s}_{i,2})$  and  $\hat{r}_i$ , since this information does not determine the values of  $e_{i,1}$  and  $e_{i,2}$ . Therefore, any point on the best line is as good as any other point

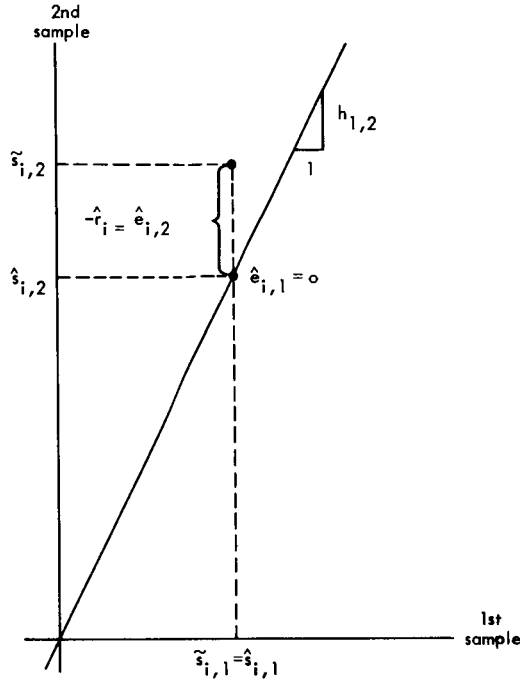


Fig. 13.

When the estimated sample point  $(\hat{s}_{i,1}, \hat{s}_{i,2})$  is chosen so that  $\hat{e}_{i,1} = 0$ , then the error  $\hat{e}_{i,2}$  is the negative of the residual  $\hat{r}_i$ . Here  $h_{1,1}$  is chosen as unity norm.

for the location of  $(\hat{s}_{i,1}, \hat{s}_{i,2})$ .

The essential difference between the method used here to determine a "best" line and the standard regression-line technique is the fact that both  $\tilde{s}_{i,1}$  and  $\tilde{s}_{i,2}$  are considered to be "noisy," whereas in the usual regression problems the true value of one of the coordinates is considered to be known exactly, and the other to be a linear function of it, but perturbed slightly by some unknown process. This perturbation is assumed to account for its displacement from the true line in a direction parallel to its coordinate axis. It is the mean-square value of this error, then, that is minimized in the usual regression problem. In this problem, however, it is a function  $h_{1,2}e_{i,1} - h_{1,1}e_{i,2}$  of the two errors whose mean square is minimized, but the result is the same.

In the special case under discussion ( $D=1$ ), it is possible to relate the mean-square residual to the mean-square error. By choosing the kernel amplitudes so that there is no error at the sample points  $\tilde{s}_{i,1}$ , that is,  $\hat{e}_{i,1} = 0$  (this can always be done), all of the error will occur at the sample points  $\tilde{s}_{i,2}$ . Then

$$\hat{r}_i = -h_{1,1}\hat{e}_{i,2} = -\hat{e}_{i,2}$$

if  $h_{1,1} = 1$ . Thus, the mean-square residual is equal to the mean-square error for this case, as shown in Fig. 13. This means that for  $D=1$  there is always a choice of kernel amplitudes for which minimizing the mean-square residual is in fact minimizing the mean-square error. Unfortunately, for  $D > 1$  such an interpretation of the residuals

is not possible. Otherwise, the extension of the ideas developed for the case  $D = 1$  to  $D > 1$  is straightforward. Instead of fitting a straight line to a set of points in two dimensions, a hyperplane is fitted to a set of points in 2D dimensions. The plane is fitted by minimizing the mean-square distance from the sample pair points to the hyperplane in a direction parallel to the axis for the samples of  $s(t)$  that are the homogeneous equation coefficients of the unknown chosen as the norm.

For an IRT of  $N$  periods, the set of homogeneous equations for span  $D$  is

$$\begin{bmatrix} S_D^N \end{bmatrix} H_D = 0. \quad (26)$$

The addition of a noise signal  $e(t)$  to the regular pulse train  $s(t)$  would cause the observed samples to deviate from the true values. The true values can be written in terms of the observed samples as in Eqs. 16. Define  $\begin{bmatrix} E_D^N \end{bmatrix}$  as the noise sample matrix

$$\begin{bmatrix} e_{1,1} & e_{1,2} & e_{2,1} & e_{2,2} & \cdot & \cdot & \cdot & e_{D,1} & e_{D,2} \\ e_{2,1} & e_{2,2} & e_{3,1} & e_{3,2} & \cdot & \cdot & \cdot & e_{D+1,1} & e_{D+1,2} \\ \cdot & & & & & & & & \\ \cdot & & & & & & & & \\ e_{N-D+1,1} & e_{N-D+1,2} & \cdot & \cdot & \cdot & \cdot & e_{N,1} & e_{N,2} \end{bmatrix}$$

Substituting Eqs. 16 in Eqs. 26 gives

$$\begin{bmatrix} \tilde{S}_D^N \end{bmatrix} H_D = \begin{bmatrix} E_D^N \end{bmatrix} H_D. \quad (27)$$

The indexing of the matrices in Eqs. 26 is based on the "period, sample" system of the IRT. For the purposes of the following discussion it is convenient to change to the conventional "row, column" system of indexing matrices. Let

$$[\tilde{U}] = \begin{bmatrix} \tilde{S}_D^N \end{bmatrix} \quad (28)$$

be an  $N - D + 1$  by  $2D$  matrix with  $u_{i,j}$  as the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. Let

$$[X] = H_D \quad (29)$$

be an  $N - D + 1$  by  $1$  matrix with  $x_i$  as the element in the  $i^{\text{th}}$  row. Let

$$[W] = \begin{bmatrix} E_D^N \end{bmatrix} \quad (30)$$

be an  $N - D + 1$  by  $2D$  matrix with  $w_{i,j}$  as the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. Equations 26 in the new notation become

$$[\tilde{U}] [X] = [W] [X]. \quad (31)$$

Defining the residuals  $\{r_i\}_{i=1}^{N-D+1}$  as

$$\left\{ r_i = \sum_{j=1}^N w_{i,j} x_j \right\}_{i=1}^{N-D+1}$$

allows Eqs. 31 to be written

$$\left\{ \sum_{j=1}^{2D} \tilde{u}_{i,j} x_j = r_i \right\}_{i=1}^{N-D+1} \quad (32)$$

As for the case of  $D = 1$ , we desire to find the  $X$  that minimizes the mean-square values of the residuals, relative to some choice of a norm. It is sufficient to minimize the sum of the squares of the residuals, since the mean can be found by dividing by  $N - D + 1$ . Then the quantity to be minimized is

$$\sum_{i=1}^{N-D+1} r_i^2 = \sum_{i=1}^{N-D+1} \left( \sum_{j=1}^{N-D+1} \tilde{u}_{i,j} x_j \right)^2. \quad (33)$$

It is convenient to defer the choice of the norm for the time being.

Setting the partials of Eqs. 33 with respect to each of the variables to zero for a minimum gives the  $2D$  equations

$$\left\{ \sum_{j=1}^{2D} \left( \sum_{i=1}^{N-D+1} \tilde{u}_{i,j} \tilde{u}_{i,k} \right) x_j = 0 \right\}_{k=1}^{2D}. \quad (34)$$

Dividing these equations by  $N - D + 1$ , the number of original equations, allows the replacing of the sums on  $i$  by averages. The equations become

$$\left\{ \sum_{j=1}^{2D} \overline{\tilde{u}_{i,j} \tilde{u}_{i,k}} x_j = 0 \right\}_{k=1}^{2D}$$

or

$$\left\{ \sum_{j=1}^{2D} c_{j,k} x_j = 0 \right\}_{k=1}^{2D}, \quad (35)$$

where  $c_{j,k} = \overline{\tilde{u}_{i,j} \tilde{u}_{i,k}}$ . These will be called the normal homogeneous equations because



they are identical to the normal equations obtained in linear regression problems.

If  $x_n$  is selected as the normed variable, then the equation for which  $k = n$  must be omitted, since there is no partial derivative with respect to  $x_n$ . Instead, the equation

$$x_n = \text{value of norm}$$

can be substituted. This gives 2D equations in 2D unknowns, which guarantees a solution.

The matrix  $[C]$  of coefficients  $c_{j,k}$  can be conveniently expressed in terms of the matrix  $[\tilde{U}]$  as

$$[C] = (N-D+1)^{-1} [\tilde{U}]^T [\tilde{U}],$$

where  $[\tilde{U}]^T$  is the transpose of  $[\tilde{U}]$ . This is easy to show. For, in general, if

$$[C] = [V][\tilde{U}]$$

for some 2D by  $N - D + 1$  matrix  $[V]$ , then

$$c_{k,j} = \sum_{i=1}^{N-D+1} v_{k,i} \tilde{u}_{i,j}.$$

Now if  $[V] = (N-D+1)^{-1} [U]^T$ , then

$$v_{k,i} = (N-D+1)^{-1} \tilde{u}_{i,k}$$

and

$$\begin{aligned} c &= (N-D+1)^{-1} \sum_{i=1}^{N-D+1} \tilde{u}_{i,k} \tilde{u}_{i,j} \\ &= \overline{\tilde{u}_{i,j} \tilde{u}_{i,k}} \end{aligned}$$

as, in fact, is the case.

#### 4.2 HOW TO FIND THE SPAN OF THE MINIMUM KERNEL IN THE PRESENCE OF NOISE

One of the difficulties to be overcome in finding the minimum kernel is that its span is not known. A strategy for determining the duration was given in section 3.5 for the ideal "noiseless" case whenever a 1-parameter acceptable solution is known to exist. It was suggested that the homogeneous equations be set up and solved for  $D = 1, 2, \dots$ , etc. until a nontrivial solution is obtained, which would then have to be the acceptable solution of minimum span.

This method cannot be used in the "noisy" case because the homogeneous equations are almost never consistent, and therefore only the trivial solution exists. The method described here of fitting a "best" solution to the homogeneous equations by solving the

normal homogeneous equations guarantees an answer, but not a solution, for any set of homogeneous equations. A strategy similar to that for the "noiseless" case could conceivably be used if there were a test whereby the correct span could be recognized when encountered.

Such a test does, in fact, exist. The test consists of solving the normal equations for span  $D = 1, 2, \dots$ , etc. and examining the behavior of the residuals obtained by substituting the solution to the normal equations back in the original homogeneous equations. When the assumed span is correct, the residuals will, in general, be noticeably smaller in magnitude. The explanation for this phenomenon is as follows.

As a preliminary notion, it must be appreciated that the rank of the coefficient matrix of the normal equations (Eqs. 34) is the same as that of the coefficient matrix of the homogeneous equations. Since we have shown that the coefficient matrix of the normal equations is the coefficient matrix of the homogeneous equations multiplied by its own transpose, the following theorem is sufficient to prove the assertion.

**THEOREM 5.** Let  $[A]$  be an  $m \times n$  matrix. Then  $\text{rank } [A]^T [A] = \text{rank } [A]$ .

**Proof:** Let  $\text{rank } [A]^T = r$ . Then the  $r^{\text{th}}$  compound<sup>9</sup> of  $[A]$ ,  $[A]^{(r)}$ , has at least one nonzero entry.

Consider the product of the compound of  $[A]^T$  with the compound of  $[A]$ . By the Binet-Cauchy theorem,<sup>9</sup>

$$[A]^{T(r)} [A]^{(r)} = ([A]^T [A])^{(r)}.$$

Since the diagonal elements of  $([A]^T [A])^{(r)}$  are the sums of the squares of the elements in the corresponding columns of  $[A]^{(r)}$ , they cannot all be zero. Hence  $\text{rank } [A]^T [A] \geq r$ . By a well-known theorem,  $\text{rank } [A]^T [A] \leq r$ . Therefore,  $\text{rank } [A]^T [A] = r$ . //

In general, the "noisy" homogeneous equations for which the "best" solution is to be found will have only the trivial solution if the number of equations is equal to or greater than the number of unknowns. That is, even though the "true" equations have a 1-, or more, parameter solution, the noisy equations do not, because the addition of the noise destroys the dependency of the equations. If the noise is small compared with the signal, however, then it might be expected that the equations are "almost dependent," a property generally known as ill-conditioned.<sup>10</sup> This property is poorly defined, but for a set of  $n$  equations in  $n$  unknowns a measure of it might be the size of the determinant of the coefficient matrix relative to its own minors. The smaller the determinant, the more ill-conditioned are the equations.

Since, by Theorem 5, the rank of the normal equations is the same as that of the homogeneous equations for the "noiseless" case, it is reasonable to expect that if the addition of noise to the homogeneous equations changes dependent equations to ill-conditioned equations, the corresponding homogeneous equations would be similarly changed.

In order to see how the test for the minimum span works, the solutions to the normal equations must be considered for three different cases: for  $D$  less than, equal to, and greater than the minimum span.

When  $D$  is less than the minimum span, the "noiseless" homogeneous equations have only the trivial solution, and so do the "noiseless" normal equations. But the normal equations are modified by deleting the equation resulting from setting the derivative with respect to the normed variable to zero, and inserting the equation

$$x_n = \text{value of the norm,}$$

where  $x_n$  is the normed variable. This procedure is based on the assumption that the "noiseless" equations have a 1-parameter solution. Since this is not the case, the equations that are solved are very different from the correct normal equations, namely the unmodified equations. Therefore the solutions will, in general, be very different from the correct solution – the trivial solution. Substitution of the solution back in the original homogeneous equations will therefore give large residuals.

When  $D$  is equal to the minimum span, it will be assumed that the "noiseless" homogeneous equations are 1-parameter. Then the "noiseless" normal equations are also 1-parameter, and the modified equations yield the correct solution if  $x_n \neq 0$ . The noisy equations are ill-conditioned, but this ill-conditioning is removed by the fixing of a norm. If the noise is not large, then the solutions that are obtained are assumed to be near the true solutions, since the equations are nearly the same. Substitution of these solutions in the original homogeneous equations should yield residuals that are much smaller in magnitude than those for lesser span because the "correct" set of normal equations was solved. A measure of the over-all magnitude is the rms value of the residuals over all the homogeneous equations for each set of 2D sample pairs, averaged over all the sets of sample pairs for the waveform. When this number shows a significant drop for some span in the sequence, then the minimum span has been encountered.

When  $D$  is greater than the minimum span, the "noiseless" homogeneous equations have a more than 1-parameter solution. The "noiseless" normal equations also have a more than 1-parameter solution. Thus the "noisy" equations will be ill-conditioned for small noise, and this ill-conditioning is not removed by modification of the equations, since only one variable is fixed. When the normal equations for a set of ill-conditioned equations are solved, they almost always have a unique solution, the only exception being the unlikely event that the equations really are dependent. By solving the ill-conditioned equations, the unknowns that could have been assigned values to eliminate the ill-conditioning effectively have values assigned to them in the solution process, and the other unknowns solved in terms of them. These "uncontrolled" solutions are useless and show up as a scattered distribution of points when they are plotted for a sequence of samples of the waveform that is being analyzed.

The solution that is obtained will give an average rms residual less than or equal to

that obtained from the equations for a lesser span. It is clear that it cannot be greater because a "best" solution to the "noisy" homogeneous equations for some span gives essentially the same residuals for equations of greater span, the extra unknowns being set to zero. Thus if a "best" solution for these equations of greater span gave a larger average rms residual than those of lesser, this solution could not be the "best" solution, since a "better" solution has been demonstrated. This is not to say that the "best" solutions for two sets of homogeneous equations of different spans (both equal to or greater than the minimum span) are likely to be nearly the same, even though the average rms residual is nearly the same. It is a property of ill-conditioned equations that solutions very different from the "best" solution may give residuals that are nearly the same. A convincing example of this situation has been given by Hartree.<sup>10</sup>

In summary, the test for the span of the "acceptable" solution of minimum span is as follows. The "best" solution to the homogeneous equations is determined for  $D = 1, 2, 3, \dots$ , etc., until a large relative drop in the average rms residual is encountered. Then the span for which the drop occurs is the "best" estimate of the minimum span. If there is no large relative drop until  $D = N$ , then either the noise is too great or the minimum span is  $N$ .

Again, we cannot say that the lowest span for which an "acceptable" solution exists is the span of the minimum kernel. As we have pointed out, we can only be sure that we have found the span of the minimum kernel if the minimum span for which there is an "acceptable" solution is the same for a large number of periodic sample pairs.

## V. ESTIMATING THE MINIMUM KERNEL AND THE IMPULSE AREAS

Our ultimate goal is to estimate the minimum kernel of a given IRT and to find a set of impulse areas that can be used to synthesize the IRT from this kernel. We shall now apply the results of our previous work to show how these objectives can be achieved.

### 5.1 ESTIMATING MANY SAMPLES OF THE MINIMUM KERNEL

We have shown how one set of coincident sample pairs of an IRT with noise can be estimated by finding the "best" solution to a set of homogeneous equations. It is now possible to estimate as many samples of the kernel as desired simply by choosing other sets of periodic sample pairs of the IRT and finding a "best" solution to the corresponding homogeneous equations. But, since we desire to normalize all of the samples of the kernel to one specified norm, it is always necessary to include the samples of the IRT that are coincident with the norm as one set of periodic samples in the chosen set of periodic sample pairs. The other set of periodic samples can be chosen at will, and each time a new selection is made the normal homogeneous equations can be solved to give a "best" estimate of the coincident samples of the kernel. The entire procedure is illustrated by the following simple example.

Let us construct an IRT with a kernel of span 2 having sample values

$$h_{1,1} = 1, h_{1,2} = 1, h_{1,3} = 0, h_{2,1} = 0, h_{2,2} = 1, h_{2,3} = 1.$$

Choose as impulse areas the numbers

$$a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 1, a_4 = -1, a_5 = 1, a_6 = 0, a_7 = 0.$$

Then the samples of the generated IRT  $s(t)$  can be found by adding. We then perturb this exact IRT by an error signal  $e(t)$  to get the final set of periodic samples of the perturbed IRT  $\tilde{s}(t)$  on the bottom line.

|                |     | 1   | 2   | 3     | 4     | 5    | 6   | 7   | 8   | Period |
|----------------|-----|-----|-----|-------|-------|------|-----|-----|-----|--------|
| Samples of     |     |     |     |       |       |      |     |     |     |        |
| $a_0 h(t+T)$   | 000 | 000 |     |       |       |      |     |     |     |        |
| $a_1 h(t)$     |     | 000 | 000 |       |       |      |     |     |     |        |
| $a_2 h(t-T)$   |     |     | 000 | 000   |       |      |     |     |     |        |
| $a_3 h(t-2T)$  |     |     |     | 110   | 011   |      |     |     |     |        |
| $a_4 h(t-3T)$  |     |     |     | -1-10 | 0-1-1 |      |     |     |     |        |
| $a_5 h(t-4T)$  |     |     |     |       | 110   | 001  |     |     |     |        |
| $a_6 h(t-5T)$  |     |     |     |       |       | 000  | 000 |     |     |        |
| $a_7 h(t-6T)$  |     |     |     |       |       |      | 000 | 000 |     |        |
| $s(t)$         |     | 000 | 000 | 110   | -101  | 10-1 | 011 | 000 | 000 |        |
| $e(t)$         |     | 000 | 000 | 0-10  | 000   | 001  | 000 | 000 | 000 |        |
| $\tilde{s}(t)$ |     | 000 | 000 | 100   | -101  | 100  | 011 | 000 | 000 |        |

We shall now try to estimate the minimum kernel by finding "best" solutions to the appropriate homogeneous equations for spans 1, 2, and 3. In each case the norm  $\hat{h}_{1,1} = 1$  will be chosen. Thus we shall choose as our two sets of periodic sample pairs the first and second samples of each period, and the first and third samples of each period. The steps in the process of solving the normal equations for each of these cases are illustrated in Figs. 14, 15, and 16. Each figure shows the coefficient matrices of the homogeneous equations  $[\tilde{S}_D^N]$ , and the normal equations whose coefficient matrices are determined by premultiplying  $[\tilde{S}_D^N]$  by its transpose, and adjusting the last row so that  $\hat{h}_{1,1} = 1$ . Also shown are the solutions to the normal equations, the residuals that are due to these solutions, and the rms residual for each set of normal equations. The average rms residual given at the bottom of the figure is taken as a measure of the "fit" of the solutions to the normal equations. This last number is 0.42, 0.27, and 0.24 for  $D = 1, 2$ , and  $3$ , respectively.

We note the large drop in the average rms residual going from  $D = 1$  to  $D = 2$ , and the subsequent relatively small decrease in this number for  $D = 3$ . We therefore conclude that  $D = 2$  is the correct span. Now a problem arises that we have not noted before. If we accept  $D = 2$  as the correct span, then we note that we have two estimates  $\hat{h}_{2,1}$ . Since we have no information about which is "better," we are free to use either estimate. Thus we shall take as our estimate of the coincident samples of the minimum kernel

$$\hat{h}_{1,1} = 1, \hat{h}_{1,2} = \frac{2}{5}, \hat{h}_{1,3} = \frac{1}{8}, \hat{h}_{2,1} = 0, \hat{h}_{2,2} = \frac{3}{5}, \hat{h}_{2,3} = \frac{7}{8}.$$

These estimated values compare only moderately well with the "true values," but it must be appreciated that the errors introduced were quite large. The experimental results of Section VII are more in keeping with what is encountered in practice.

## 5.2 FINDING THE IMPULSE AREAS

Since it may sometimes be desirable to find the impulse areas corresponding to an estimated kernel, we propose a method for finding them, although no examples of its use are given.

In the noiseless case, finding the impulse areas for just one set of periodic sample pairs of an IRT and a given acceptable solution to the homogeneous equations is just a matter of solving the generating equations (Eqs. 8) for all  $p$ . Since the coefficients are acceptable solutions to the corresponding homogeneous equations, a solution to these generating equations is guaranteed.

In a practical case, the samples of the waveform to be analyzed are not likely to be exactly the samples of an IRT. Also, the coefficients of the corresponding generating equations will be only the estimated samples of the kernel as found by solving the normal equations. Therefore, an exact solution to the generating equations will almost never exist. So a "best" estimate of the impulse areas must be obtained. The "best"

|  | COEFFICIENT MATRIX<br>OF THE HOMOGENEOUS<br>EQUATIONS               | NORMAL<br>EQUATIONS  | SOLUTIONS TO<br>NORMAL<br>EQUATIONS   | RESIDUALS   |
|--|---|--|---|---|
| 1ST AND 2ND<br>SAMPLES OF EACH<br>PERIOD OF $\tilde{s}(t)$ | $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$  | $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_{1,2} \\ \hat{h}_{1,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ | $\begin{aligned} \hat{h}_{1,2} &= 0 \\ \hat{h}_{1,1} &= 1 \end{aligned}$<br><br>rms RESIDUAL    | $\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \\ r_3 &= 0 \\ r_4 &= -1 \\ &= 0.25 \end{aligned}$          |
| 1ST AND 3RD<br>SAMPLES OF EACH<br>PERIOD OF $\tilde{s}(t)$ | $\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_{1,3} \\ \hat{h}_{1,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ | $\begin{aligned} \hat{h}_{1,3} &= -1/3 \\ \hat{h}_{1,1} &= 1 \end{aligned}$<br><br>rms RESIDUAL | $\begin{aligned} r_1 &= -1/3 \\ r_2 &= -2/3 \\ r_3 &= -1/3 \\ r_4 &= -1 \\ &= 0.42 \end{aligned}$ |
| AVERAGE rms RESIDUAL = $\frac{0.25 + 0.42}{2} = 0.35$      |   |  |   |   |

Fig. 14. Estimating the samples of the kernel for span 1.

|  | COEFFICIENT MATRIX<br>OF THE HOMOGENEOUS<br>EQUATIONS   | NORMAL<br>EQUATIONS   | SOLUTIONS TO<br>NORMAL<br>EQUATIONS   | RESIDUALS   |
|--|---|---|---|---|
| 1ST AND 2ND<br>SAMPLES OF EACH<br>PERIOD OF $\tilde{s}(t)$ | $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$   | $\begin{bmatrix} 3 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_{2,2} \\ \hat{h}_{2,1} \\ \hat{h}_{1,2} \\ \hat{h}_{1,1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$   | $\begin{aligned} \hat{h}_{2,2} &= 3/5 \\ \hat{h}_{2,1} &= 0 \\ \hat{h}_{1,2} &= 2/5 \\ \hat{h}_{1,1} &= 1 \end{aligned}$    | $\begin{aligned} r_1 &= 2/5 \\ r_2 &= 1/5 \\ r_3 &= -1 \\ r_4 &= -2 \\ r_5 &= 0 \end{aligned}$        |
|  |   |   | rms RESIDUAL = 0.27   |   |
| 1ST AND 3RD<br>SAMPLES OF EACH<br>PERIOD OF $\tilde{s}(t)$ | $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 3 & 1 & -2 & -2 \\ 1 & 2 & -1 & 0 \\ -2 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_{2,3} \\ \hat{h}_{2,1} \\ \hat{h}_{1,3} \\ \hat{h}_{1,1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ | $\begin{aligned} \hat{h}_{2,3} &= 7/8 \\ \hat{h}_{2,1} &= -3/8 \\ \hat{h}_{1,3} &= 1/8 \\ \hat{h}_{1,1} &= 1 \end{aligned}$ | $\begin{aligned} r_1 &= 1/8 \\ r_2 &= -2/8 \\ r_3 &= -3/8 \\ r_4 &= -1/8 \\ r_5 &= 3/8 \end{aligned}$ |
|  | AVERAGE rms RESIDUAL = 0.27   |   | rms RESIDUAL = 0.27   |   |

Fig. 15. Estimating the samples of the kernel for span 2.



|  | COEFFICIENT MATRIX<br>OF THE HOMOGENEOUS<br>EQUATIONS   | NORMAL<br>EQUATIONS   | SOLUTIONS TO<br>NORMAL<br>EQUATIONS  | RESIDUALS  |
|--|---|---|--|--|
| 1ST AND 2ND<br>SAMPLES OF EACH<br>PERIOD OF $\tilde{z}(t)$ | $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$    | $\begin{bmatrix} 3 & 0 & -2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 3 & 0 & -2 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$      | $\begin{aligned} \hat{h}_{3,2} &= -1/3 \\ \hat{h}_{3,1} &= 0 \\ \hat{h}_{2,2} &= 1/3 \\ \hat{h}_{2,1} &= -1/3 \\ \hat{h}_{1,2} &= 1/3 \\ \hat{h}_{1,1} &= 1 \end{aligned}$ | $\begin{aligned} r_1 &= 1/3 \\ r_2 &= 0 \\ r_3 &= -1/3 \\ r_4 &= -1/3 \\ r_5 &= 0 \\ r_6 &= 0 \end{aligned}$ |
|  |   |   | rms RESIDUAL = 0.24  |  |
| 1ST AND 3RD<br>SAMPLES OF EACH<br>PERIOD OF $\tilde{z}(t)$ | $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 3 & 1 & -2 & -2 & 1 & 1 \\ 1 & 2 & -1 & 0 & 0 & 1 \\ -2 & -1 & 3 & 1 & -2 & -2 \\ -2 & 0 & 1 & 2 & -1 & 0 \\ 1 & 0 & -2 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ | $\begin{aligned} \hat{h}_{3,3} &= -1/3 \\ \hat{h}_{3,1} &= 0 \\ \hat{h}_{2,3} &= 2/3 \\ \hat{h}_{2,1} &= -2/3 \\ \hat{h}_{1,3} &= 0 \\ \hat{h}_{1,1} &= 1 \end{aligned}$   | $\begin{aligned} r_1 &= 0 \\ r_2 &= -1/3 \\ r_3 &= -1/3 \\ r_4 &= 0 \\ r_5 &= 1/3 \\ r_6 &= 0 \end{aligned}$ |
|  | AVERAGE rms RESIDUAL = 0.24   |   | rms RESIDUAL = 0.24  |  |

Fig. 16. Estimating the samples of the kernel for span 3.

estimate will be those impulse areas that minimize the mean-square error between the samples of the IRT that they generate and the samples of the waveform that is being analyzed.

When the number of impulse areas to be found is small, the "best" impulse areas can be found by solving the set of normal equations corresponding to the generating equations. The normal equations corresponding to an arbitrary set of linear equations can be derived as follows.

Consider the following set of linear equations.

$$\left\{ \sum_{j=1}^J c_{ij} x_j = y_i \right\}_{i=1}^I \quad I \geq J.$$

Find the set of numbers  $\{x_j\}_{j=1}^J$  that minimize

$$\sum_{i=1}^I \left( y_i - \sum_{j=1}^J c_{ij} x_j \right)^2.$$

Setting the partial with respect to  $x_k$  to zero gives the normal equation

$$\sum_{i=1}^I \left( c_{ik} y_i - \sum_{j=1}^J c_{ik} c_{ij} x_j \right) = 0$$

or

$$\sum_{j=1}^J \left( \sum_{i=1}^I c_{ik} c_{ij} \right) x_j = \sum_{i=1}^I c_{ik} y_i.$$

The entire set of normal equations obtained in this way is

$$\left\{ \sum_{j=1}^J \left( \sum_{i=1}^I c_{ik} c_{ij} \right) x_j = \sum_{i=1}^I c_{ik} y_i \right\}_{k=1}^J.$$

In matrix notation these equations can be written

$$[C]^T [C] X = [C]^T Y.$$

For an entire set of homogeneous equations, the normal equations in matrix notation would be

$$\left[ \hat{H}_D^{N, M} \right]^T \left[ \hat{H}_D^{N, M} \right] A_D^N = \left[ \hat{H}_D^{N, M} \right]^T \tilde{S}_D^{N, M}.$$

The notation of section 5.1 is combined with the circumflex and tilde to denote estimated and noisy values, respectively.

When the number of impulse areas to be found is large, the round-off errors involved in solving the normal equations for a large number of unknowns may cause large errors in the solutions. In this eventuality, an iterative procedure would be preferable if the convergence is sufficiently rapid. By taking advantage of the particular arrangement of entries in the coefficient matrix of the generating equations, a useful iterative procedure can be developed. Each new iteration is guaranteed to be better than the last, and the first iteration should be "close."

The method is the following.

(i) Choose a set of initial values for the unknowns in the following way. Find the initial estimate of the unknown  $a_i$  by solving the normal equations corresponding to the subset of generating equations in which  $a_i$  appears as an unknown. These normal equations will have, at most,  $2D-1$  unknowns.

(ii) With all the other unknowns fixed at their values as determined in (i), choose as the next estimate of the first unknown the value that minimizes the mean-square error for all the waveform samples to which it contributes. Repeat this procedure for each succeeding unknown. Each step involves solving the normal equation for DM (or less) equations in one unknown. The new values selected in this way must be at least as good as those determined in (i), since the value determined by the normal equation is the one that minimizes the error, that is, the "best" value; therefore the previous value cannot be better.

(iii) Repeat (ii) for all the unknowns until the mean-square error between the generated and the actual values is no longer reduced significantly.

## VI. SUGGESTIONS FOR CHOOSING THE NORM AND FINDING THE IMPULSE SPACING

### 6.1 CHOOSING THE NORM

When the homogeneous equations are solved to find the periodic sample pairs, one of the unknown samples must be designated as norm, and assigned an arbitrary value. This value serves as a scale factor for all of the other unknowns when the solution is 1-parameter. It does not matter which unknown is chosen as norm, as long as it is not constrained to be zero by the equations themselves.

Consider, for example, the homogeneous equations

$$\begin{aligned} h_{1,2} &= 0 \\ h_{2,1} - h_{1,2} - h_{1,1} &= 0 \\ -h_{3,1} - h_{3,2} + h_{1,1} &= 0 \\ h_{4,2} &= 0. \end{aligned}$$

The general solution to this set of equations is  $h_{1,1} = h_{2,2}$ ,  $h_{1,2} = h_{2,1} = 0$ . If  $h_{1,1}$  or  $h_{2,2}$  are given fixed values, then the equations can be solved. If  $h_{1,2}$  or  $h_{2,1}$  are assigned nonzero values, the resulting equations do not have a solution because the rank of the coefficient matrix will be less than the rank of the augmented matrix.<sup>6</sup> Since the homogeneous equations do not have a solution in the noiseless case, the normal equations will not yield a "best" solution in the noisy case. The remedy is to choose a different norm. Since we are primarily interested in the case in which the beginning of the waveform is known, a norm can always be chosen from the "exposed" first period with assurance that it is nonzero.

### 6.2 FINDING THE IMPULSE SPACING

A general method for finding the impulse spacing, or period as we have called it, is unknown to the author. It is possible, however, to make some suggestions that are useful for some types of IRT.

(i) Direct examination of the waveform. This rather obvious method can always be used when the kernel amplitudes are constant or slowly varying over a number of periods substantially greater than the span of the kernel. Within this interval the IRT is a periodic or quasi-periodic function, and the period is equal to the spacing of the impulses. Many physical signals that can be modeled as IRT exhibit such a "steady-state" region from which the period can be determined.

Direct examination of the waveform may also reveal the period if the kernel has relatively high sharp peaks that show up in the IRT at one-period intervals. In any case, the waveform should be examined for any repetitive features that would suggest the period.

(ii) Examination of the autocorrelation function. As can be easily shown, the autocorrelation function of an IRT is itself an IRT whose kernel is the autocorrelation function of the kernel of the original train. The autocorrelation function of

$$s(t) = \sum_{n=1}^N a_n h(t-nT)$$

is

$$\begin{aligned} SS(\sigma) &= \int_{-\infty}^{\infty} s(t) s(t+\sigma) dt \\ &= \int_{-\infty}^{\infty} \sum_{n=1}^N a_n h(t-nT) \sum_{m=1}^N a_m h(t-mT+\sigma) dt \\ &= \sum_{n=1}^N \sum_{m=1}^N a_n a_m \int_{-\infty}^{\infty} h(t-nT) h(t-mT+\sigma) dt. \end{aligned}$$

Putting  $u = t - nT$  gives

$$\begin{aligned} SS(\sigma) &= \sum_{n=1}^N \sum_{m=1}^N a_n a_m \int_{-\infty}^{\infty} h(u) h(u+(n-m)T+\sigma) dt \\ &= \sum_{n=1}^N \sum_{m=1}^N a_n a_m HH(\sigma-(m-n)T), \end{aligned}$$

where

$$HH(\sigma) = \int_{-\infty}^{\infty} h(t) h(t+\sigma) dt.$$

If the kernel is weakly correlated with itself for displacements greater than  $T/2$ , then the autocorrelation function of the IRT will be a series of pulses of span  $D = 1$ , and the period will be easily detected.

(iii) Examination of the spectrum. Since an IRT has features recurring at regular intervals, it seems likely that the spectrum of the pulse train would have a peak at the frequency corresponding to the period. The spectrum shows some other characteristic features also. The Fourier transform of

$$s(t) = \sum_{n=1}^N a_n h(t-nT)$$

is

$$S(j\omega) = H(j\omega) \sum_{n=1}^N a_n e^{-jnT\omega}.$$

The spectrum of an IRT is a periodic function in  $\omega$ , with "period"  $2\pi/T$ , that is, "amplitude-modulated" by the spectrum  $H(j\omega)$  of  $h(t)$ . This characteristic "amplitude modulation" may be difficult to detect, but if it should be apparent for a given waveform, then the period of the IRT can be computed. This method is only likely to be useful if the spectrum of  $H(j\omega)$  varies slowly over the interval  $2\pi/T$ , and stretches over many such intervals. Unfortunately, such a spectrum usually means that the pulse  $h(t)$  is short compared with the time interval  $T$ , in which case the period, as well as the kernel itself, will be quite obvious from the waveform.

(iv) Correlation with a sine wave. Correlation of  $s(t)$  with a sine wave to determine the period of the IRT is exactly the same as looking for a peak in the spectrum of  $s(t)$ . Since the spectrum can often be easily obtained, there is no reason to use this method. There is no guarantee that this method will work because of the drastic influence that the spectrum of  $h(t)$  may have on the spectrum of the IRT. Also, sharp peaks in the spectrum mean that the time function has periodicities, which are usually easy to detect by direct examination of the waveform.

## VII. EXPERIMENTAL RESULTS

The techniques that have been developed will now be applied to particular examples. Three different examples of the extraction of the minimum kernel are presented, and their properties noted.

### 7.1 EXAMPLE 1

Figure 17 is an oscilloscope trace of the response of an RLC circuit to a short rectangular excitation pulse. The trace of this excitation pulse is superimposed on that



Fig. 17. Actual kernel of the IRT of Example 1 (0.4 msec/cm).

of the response. By exciting the circuit at regular intervals with excitations of this same rectangular shape but different amplitudes, the IRT shown in Fig. 18 was produced (the waveform is zero outside the interval shown in the photographs). The excitations are superimposed for reference, and show the period.

Samples of this IRT were taken directly from the photographs, at intervals corresponding to the smallest graticule division. The measurements were made to the nearest half-division of the smallest graticule division. The corresponding homogeneous equations were then solved under the assumption of span of 1-5 periods, and the average rms residual computed for each span. The average rms residual is plotted as a function of duration in Fig. 19. The graph shows a large decrease going from a duration of 1 period to a duration of 2 periods, thereby indicating that 2 is the "correct" span, that is, the span of the minimum kernel.

In Fig. 20, the solutions to the homogeneous equations, as computed for spans of 1, 2, and 3, are plotted superimposed on the actual kernel for comparison. The following features are of interest.

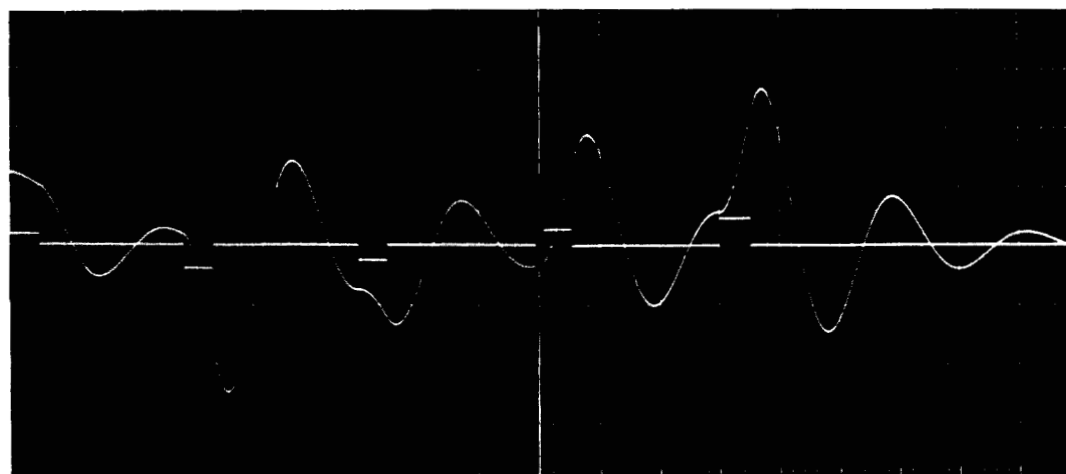
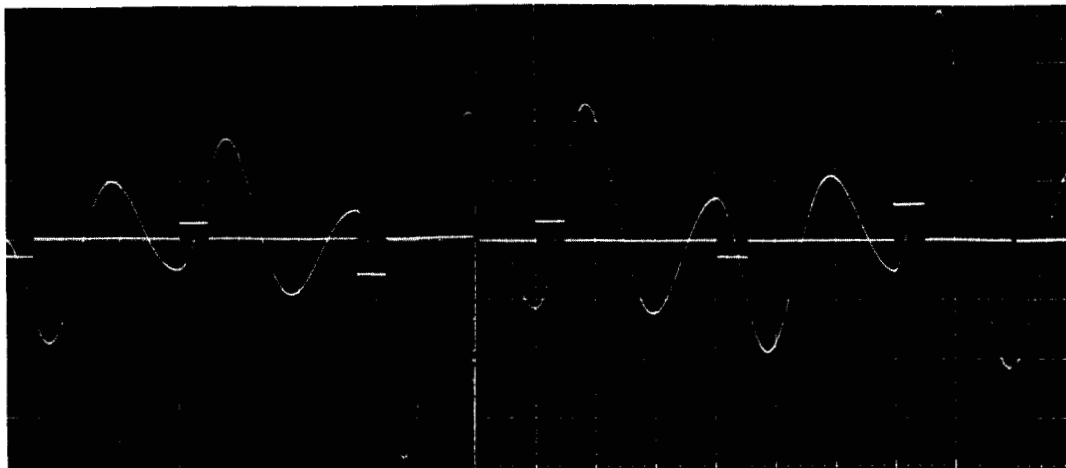


Fig. 18. The IRT of Example 1 (0.4 msec/cm). The bottom trace continues from the top trace.

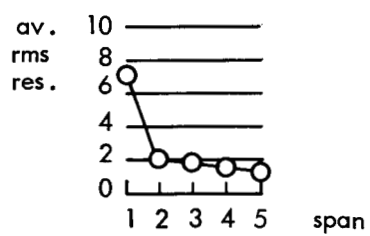


Fig. 19. Average rms residual as a function of span for Example 1.



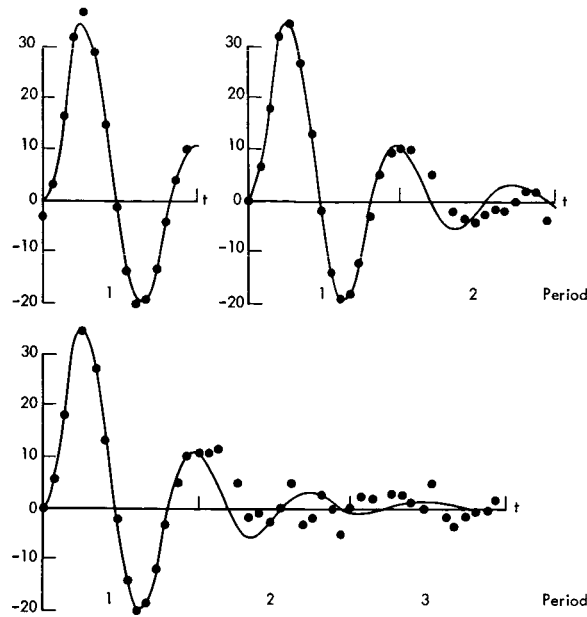


Fig. 20. Solutions of the homogeneous equations compared with the waveform of the actual kernel for Example 1.

(i) The actual kernel is the minimum kernel. The actual kernel is essentially zero after two periods, that is, it has span 2, and the solutions are very close to the waveform of the actual kernel for span 2.

(ii) The solutions for span 1 lie very close to the waveform of the actual span in the first period. This feature will appear in the other examples, but no satisfactory explanation can be given for the phenomenon.

(iii) For span 3 the solutions become "erratic" after the first period, no longer tracing out a continuous curve. This feature will be more noticeable in the other examples. This behavior is due to the ill-conditioning of the equations for spans greater than the minimum, as predicted in Section IV. The "fit" in the first period, however, appears to be very good.

## 7.2 EXAMPLE 2

The IRT of this example was supplied by Professor J. S. MacDonald. All of the computations were made by the author without foreknowledge of the actual kernel.

Figure 21 is an oscilloscope trace of the actual kernel which generated the IRT of Fig. 22. Figures 21a and 22a show the complete waveforms; Figs. 21b and 22b show the same waveforms with expanded time scales. The zero amplitude line is the one on which the 2-cm radius circles appear on the graticule. The waveforms are essentially zero outside the intervals shown in the figures. The generating pulse waveform of Fig. 21a is actually the response of a linear system to a relatively short rectangular exciting pulse, and the regular pulse train was generated by exciting this linear system

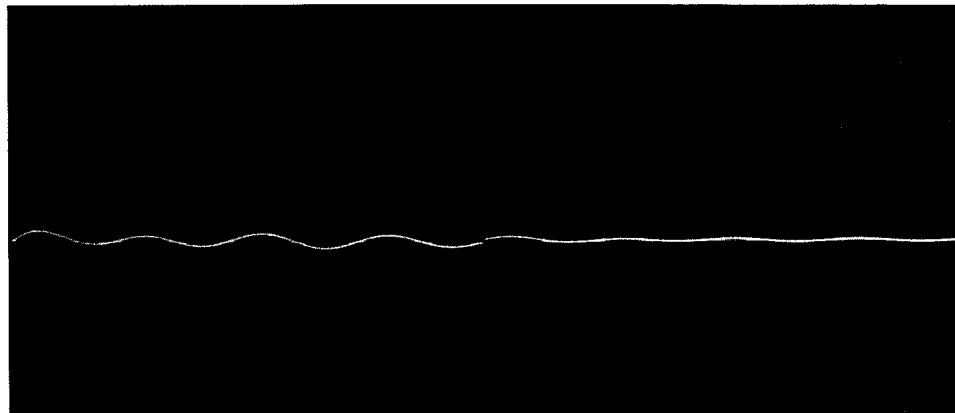
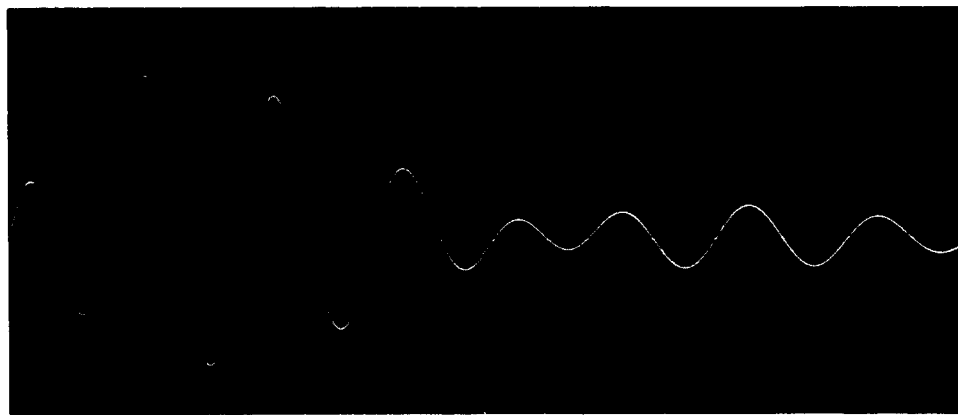
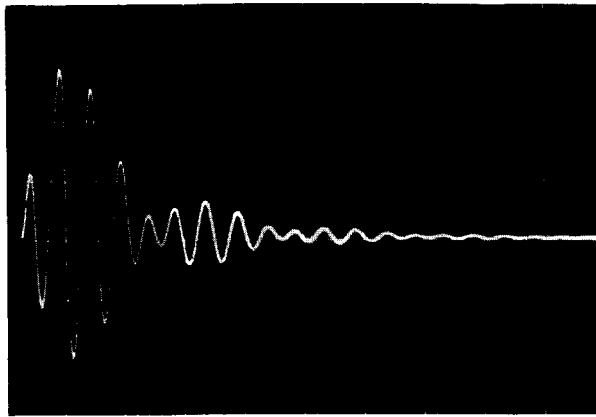


Fig. 21. (a) Actual kernel of the IRT of Example 2 (2 msec/cm).  
 (b) Actual kernel of the IRT of Example 2 (0.5 msec/cm).  
 The bottom trace continues from the top trace.

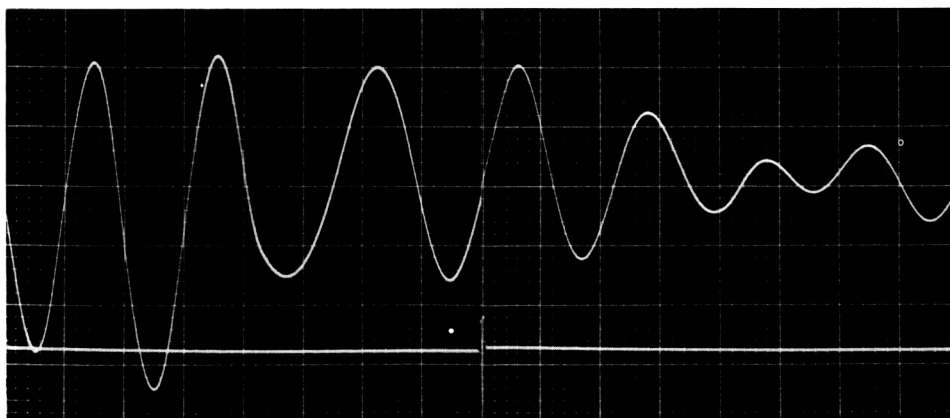
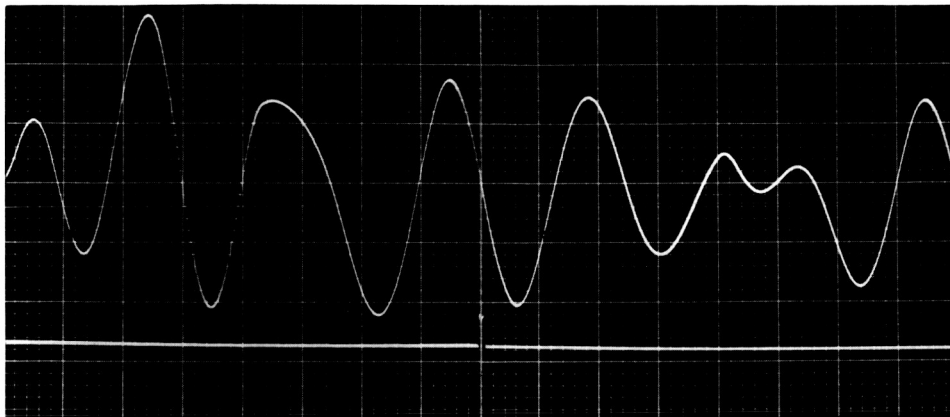
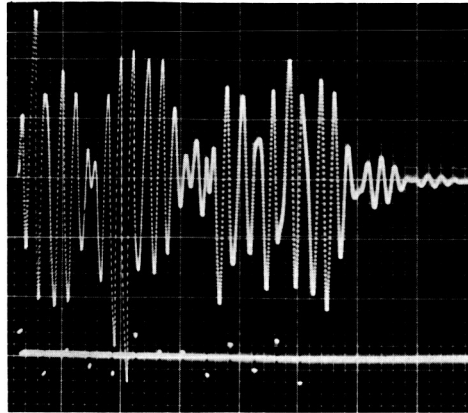


Fig. 22. (a) The IRT of Example 2 (5 msec/cm).  
 (b) The IRT of Example 2 (0.5 msec/cm).  
 The bottom trace continues from the  
 top trace.

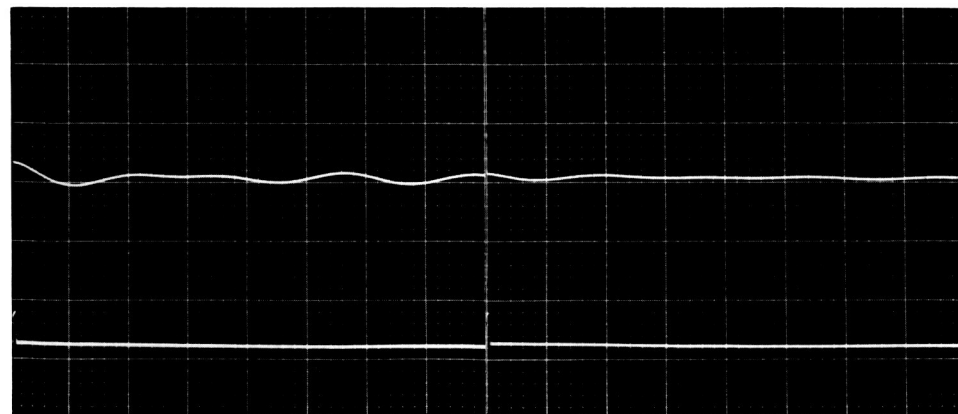
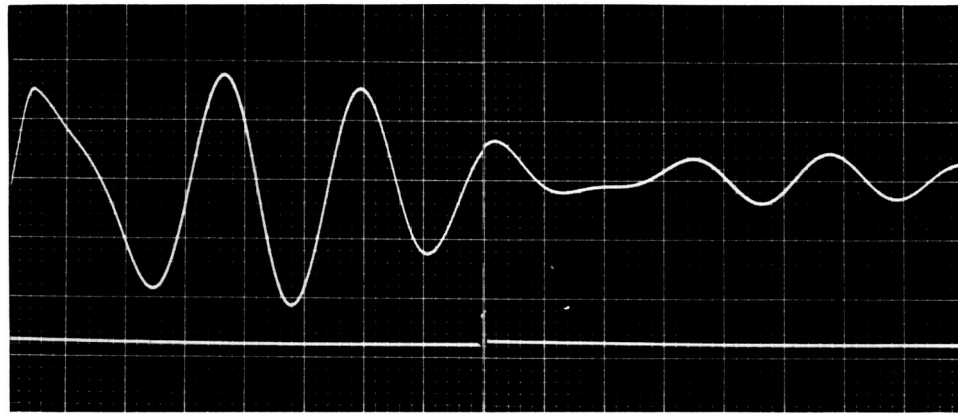
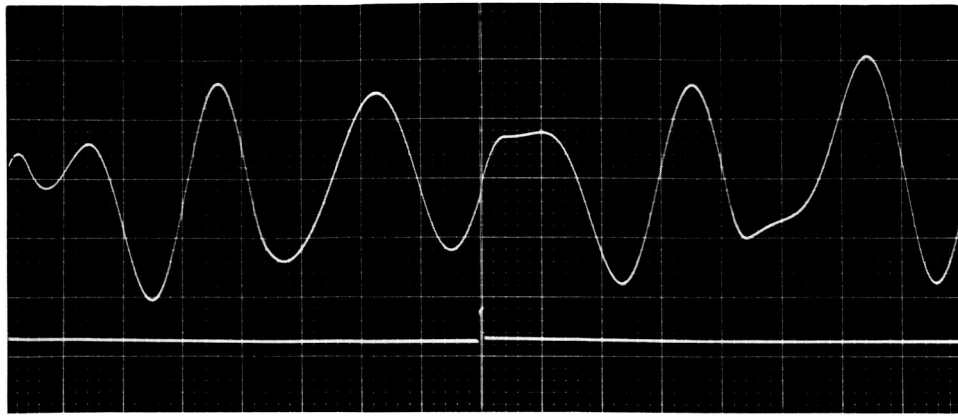


Fig. 22. Continued.

with a sequence of these pulses occurring at regular intervals. These exciting pulses are shown in the bottom trace of Fig. 22a. The trace at the bottom of Fig. 22b can be ignored. The period of the IRT in Fig. 22b is 2 msec (4 cm on the figure).

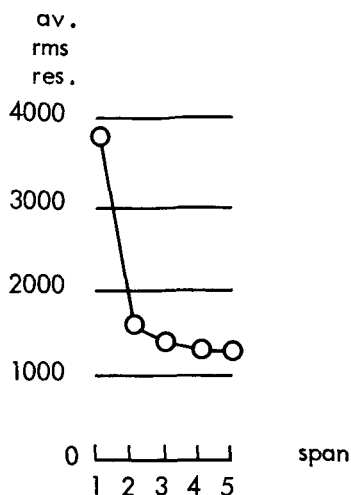


Fig. 23. Average rms residual as a function of span for Example 2.

Samples of this IRT were taken directly from the photographs at intervals corresponding to the smallest graticule division (20 samples per period). The measurements were made to the nearest half-division of the smallest graticule division. The corresponding homogeneous equations were then solved for spans of 1-5 periods, and the average rms residual computed for each span, and plotted in Fig. 23. This graph shows that 2 is the "correct" span.

Figure 24 compares the solutions for span 1, 2, and 3 with the waveform of the actual kernel. The following features are of interest.

(i) The actual kernel is not the minimum kernel, since the span of the actual kernel is certainly greater than 2. Figure 23 shows that the estimated samples of the kernel of minimum span 2 are nearly coincident with the first two periods of the actual kernel. Examination of the actual kernel shows that except for a scale factor it very nearly repeats itself every two periods.

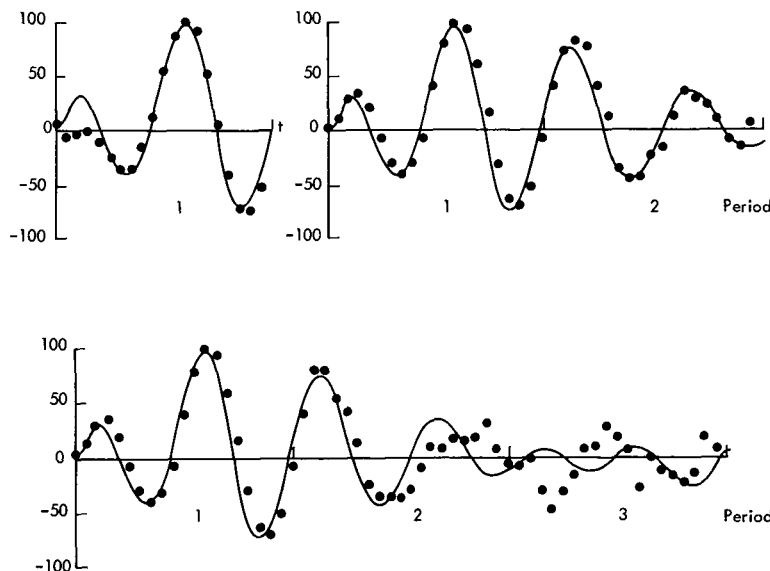


Fig. 24. Solutions to the homogeneous equations compared with the waveform of the actual kernel for Example 2.

Thus the actual kernel is itself an IRT whose kernel is the estimated minimum kernel. Thus this estimated kernel is capable of generating any IRT generated by the kernel of Fig. 21.

(ii) As in the previous example, the solutions for span 1 lie close to the waveform of the actual kernel in the first period.

(iii) For span 3 the solutions show a marked "erratic" behavior after about the middle of the second period. As before, this behavior is believed to be due to the ill-conditioning of the normal equations.

### 7.3 EXAMPLE 3

The IRT of this example was generated artificially on a computer, and hence all the samples are accurate to approximately 8 figures. The actual kernel was the function

$$t(t-13)(t-31)e^{-0.10t} \quad 0 \leq t \leq 200$$

This function is plotted in Fig. 25. The period was chosen to be 20 time units, and

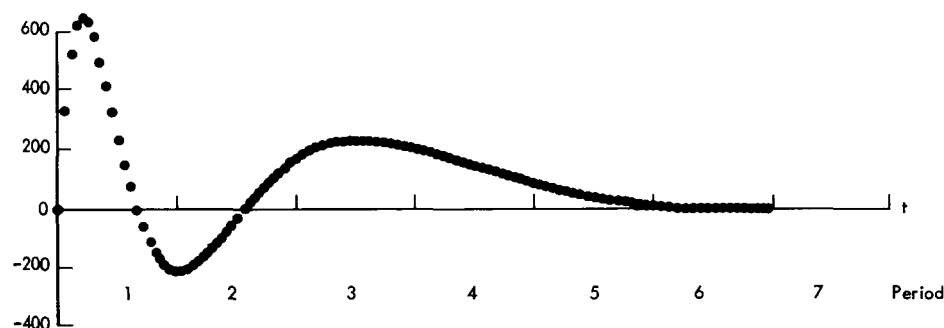


Fig. 25. Actual kernel for the IRT of Example 3.

the impulse areas selected from a table of random numbers (except for the zero amplitudes, included to make the IRT complete). These impulse areas are listed in Table 1. The samples of the resulting IRT taken every time unit (20 samples per period) are listed in Table 2.

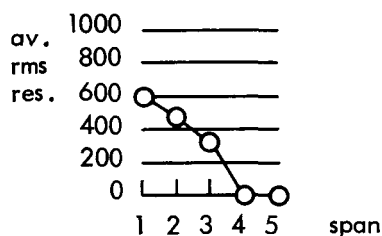


Fig. 26. Average rms residual as a function of span for Example 3.

The homogeneous equations for these samples were solved for spans 1-5, and the average rms residual computed and plotted for each span. This plot indicates rather dramatically that 4 is the "correct" span. The solutions for spans 1-5 are plotted in Fig. 27. The following features may be observed.

(i) The actual kernel is not the minimum kernel. Furthermore, the actual

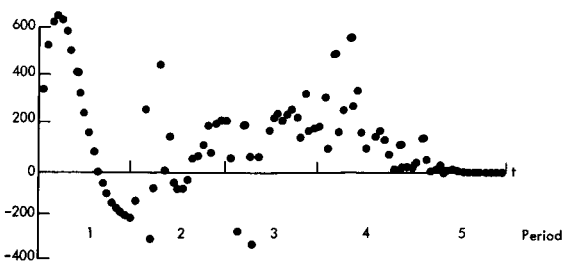
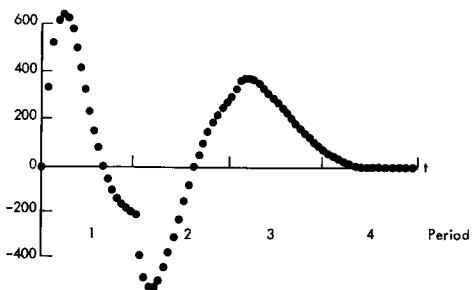
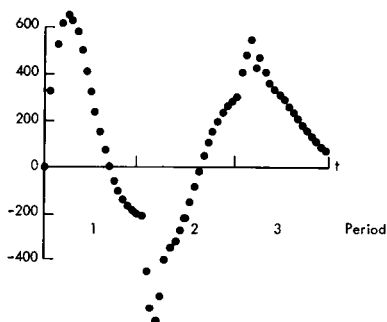
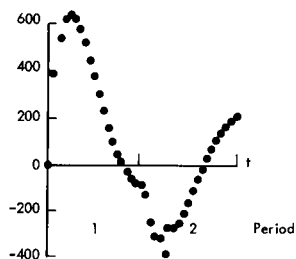
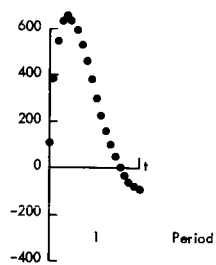


Fig. 27. Solutions to the homogeneous equations for Example 3.

kernel does not appear to be a scaled repetition of the minimum kernel.

(ii) All five solutions match the actual kernel very closely in the first period.

(iii) The solutions show a very pronounced "erratic" behavior for span 5. Apparently the accuracy of the samples caused the normal equations to become very nearly dependent.

Table 1. Impulse areas for the IRT of Example 3.

Read from left to right and down the page. The number following the E is the power of 10 by which the decimal fraction is to be multiplied. (4E15.8 format)

|                |                |                |                |
|----------------|----------------|----------------|----------------|
| +.00000000E+01 | +.00000000E+01 | +.00000000E+01 | +.00000000E+01 |
| +.00000000E+01 | +.00000000E+01 | +.00000000E+01 | +.00000000E+01 |
| -.53479000E+01 | +.97344000E+01 | -.66023000E+01 | +.99760000E+01 |
| +.30176000E+01 | +.81874000E+01 | -.19839000E+01 | -.09377000E+01 |
| -.31151000E+01 | -.67619000E+01 | +.61946000E+01 | -.04811000E+01 |
| -.05763000E+01 | +.73260000E+01 | -.54909000E+01 | -.42583000E+01 |
| +.27266000E+01 | -.49843000E+01 | +.29316000E+01 | -.30463000E+01 |
| +.28708000E+01 | -.13183000E+01 | +.60796000E+01 | +.13486000E+01 |
| +.34914000E+01 | -.28105000E+01 | -.59231000E+01 | -.87437000E+01 |
| +.29046000E+01 | -.62035000E+01 | +.38856000E+01 | +.40666000E+01 |
| +.40588000E+01 | -.78237000E+01 | -.98247000E+01 | -.69977000E+01 |
| -.39843000E+01 | +.62880000E+01 | +.56138000E+01 | +.90804000E+01 |
| +.00000000E+01 | +.00000000E+01 | +.00000000E+01 | +.00000000E+01 |

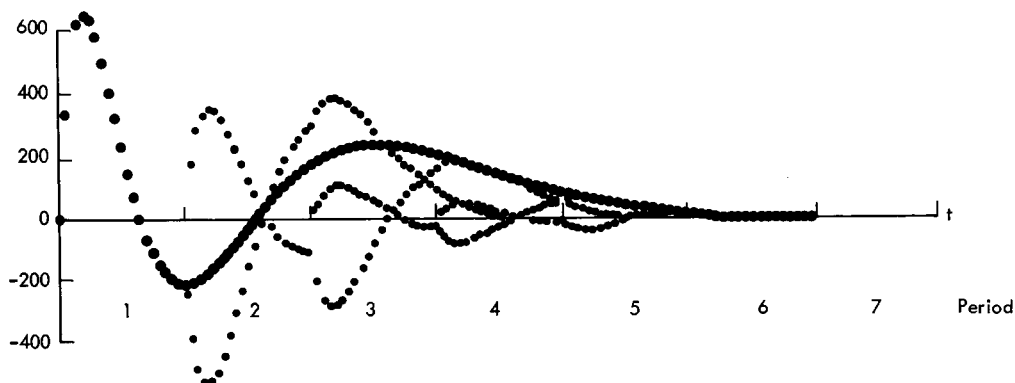


Fig. 28. Actual kernel as an IRT generated by the minimum kernel.

It is not necessary to reconstruct the IRT from the estimated minimum kernel to be certain that it really is a kernel for the given IRT. Figure 28 is a demonstration of



Table 2. Samples of the IRT of Example 3.

Read from left to right and down the page. The number following the E is the power of 10 by which the decimal fraction is to be multiplied. (4E15.8 format) Table is continued on the next four pages.

There are 20 samples per period.

[illegible]

|             |    |             |    |             |    |             |    |
|-------------|----|-------------|----|-------------|----|-------------|----|
| .00000000E  | 00 | -.17420328E | 04 | -.27934767E | 04 | -.33279302E | 04 |
| -.34844300E | 04 | -.33734119E | 04 | -.30817392E | 04 | -.26769341E | 04 |
| -.22107291E | 04 | -.17220408E | 04 | -.12394510E | 04 | -.78327098E | 03 |
| -.36725250E | 03 | .00000000E  | 00 | .31386868E  | 03 | .57277337E  | 03 |
| .77740017E  | 03 | .93007859E  | 03 | .10342823E  | 04 | .10942332E  | 04 |
| .11145898E  | 04 | .42711041E  | 04 | .61407153E  | 04 | .70441542E  | 04 |
| .72390170E  | 04 | .69305488E  | 04 | .62807506E  | 04 | .54160572E  | 04 |
| .44337918E  | 04 | .34075798E  | 04 | .23918747E  | 04 | .14257322E  | 04 |
| .53594419E  | 03 | -.26036672E | 03 | -.95361032E | 03 | -.15399757E | 04 |
| -.20200016E | 04 | -.23974182E | 04 | -.26782140E | 04 | -.28698926E | 04 |
| -.29808843E | 04 | -.51707313E | 04 | -.64452330E | 04 | -.70275735E | 04 |
| -.70979243E | 04 | -.68006661E | 04 | -.62505041E | 04 | -.55376367E | 04 |
| -.47321118E | 04 | -.38874938E | 04 | -.30439366E | 04 | -.22307566E | 04 |
| -.14685762E | 04 | -.77110699E | 03 | -.14662320E | 03 | .40082205E  | 03 |
| .87040037E  | 03 | .12636952E  | 04 | .15840430E  | 04 | .18360037E  | 04 |
| .20249377E  | 04 | .54062703E  | 04 | .74482116E  | 04 | .84806381E  | 04 |
| .87688125E  | 04 | .85243957E  | 04 | .79147489E  | 04 | .70707778E  | 04 |
| .60935261E  | 04 | .50597107E  | 04 | .40263481E  | 04 | .30346131E  | 04 |
| .21130473E  | 04 | .12802129E  | 04 | .54688250E  | 03 | -.82165481E | 02 |
| -.60668052E | 03 | -.10297508E | 04 | -.13568452E | 04 | -.15950482E | 04 |
| -.17524540E | 04 | -.85473718E | 03 | -.28332867E | 03 | .51025348E  | 02 |
| .21838361E  | 03 | .27311828E  | 03 | .25684869E  | 03 | .20069058E  | 03 |
| .12829029E  | 03 | .55512310E  | 02 | -.61705169E | 01 | -.49532470E | 02 |
| -.70461074E | 02 | -.67196615E | 02 | -.39718096E | 02 | .10749052E  | 02 |
| .82124797E  | 02 | .17178257E  | 03 | .27678868E  | 03 | .39408609E  | 03 |
| .52063128E  | 03 | .33204716E  | 04 | .50666262E  | 04 | .60223563E  | 04 |
| .63982701E  | 04 | .63614649E  | 04 | .60432269E  | 04 | .55455033E  | 04 |
| .49463293E  | 04 | .43043639E  | 04 | .36626698E  | 04 | .30518536E  | 04 |
| .24926641E  | 04 | .19981333E  | 04 | .15753326E  | 04 | .12268056E  | 04 |
| .95173091E  | 03 | .74685702E  | 03 | .60725169E  | 03 | .52689341E  | 03 |
| .49913446E  | 03 | -.12918053E | 03 | -.46254057E | 03 | -.57205790E | 03 |
| -.51567698E | 03 | -.34028844E | 03 | -.83533084E | 02 | .22466306E  | 03 |
| .56078113E  | 03 | .90660322E  | 03 | .12482730E  | 04 | .15755042E  | 04 |
| .18809140E  | 04 | .21594615E  | 04 | .24079806E  | 04 | .26247876E  | 04 |
| .28093569E  | 04 | .29620525E  | 04 | .30839062E  | 04 | .31764380E  | 04 |
| .32415079E  | 04 | .29757508E  | 04 | .28079119E  | 04 | .27098155E  | 04 |
| .26593408E  | 04 | .26393325E  | 04 | .26366853E  | 04 | .26415794E  | 04 |
| .26468398E  | 04 | .26474065E  | 04 | .26398916E  | 04 | .26222175E  | 04 |
| .25933179E  | 04 | .25528924E  | 04 | .25012102E  | 04 | .24389480E  | 04 |
| .23670638E  | 04 | .22866946E  | 04 | .21990765E  | 04 | .21054843E  | 04 |
| .20071842E  | 04 | .89068221E  | 03 | .17411276E  | 03 | -.24257231E | 03 |
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| -.10690264E | 03 | .81808007E  | 02 | .27045126E  | 03 | .44750752E  | 03 |
| .60520407E  | 03 | .73869926E  | 03 | .84541408E  | 03 | .92448562E  | 03 |
| .97632501E  | 03 | .10022604E  | 04 | .10042520E  | 04 | .96466704E  | 03 |
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| -.37503833E | 04 | -.37042712E | 04 | -.34339119E | 04 | -.30228063E | 04 |
| -.25347065E | 04 | -.20173733E | 04 | -.15057061E | 04 | -.10243432E | 04 |
| -.58981174E | 03 | -.21229943E | 03 | .10289260E  | 03 | .35416535E  | 03 |
| .54266179E  | 03 | .67149379E  | 03 | .74512507E  | 03 | .76888329E  | 03 |
| .74864853E  | 03 | .27081135E  | 04 | .38355576E  | 04 | .43378348E  | 04 |
| .43814525E  | 04 | .40993969E  | 04 | .35969089E  | 04 | .29563586E  | 04 |
| .22413522E  | 04 | .15001821E  | 04 | .76871659E  | 03 | .72814699E  | 02 |
| -.56966403E | 03 | -.11471922E | 04 | -.16531974E | 04 | -.20849292E | 04 |
| -.24425328E | 04 | -.27282957E | 04 | -.29460384E | 04 | -.31006271E | 04 |
| -.31977129E | 04 | -.33996250E | 04 | -.34936590E | 04 | -.35013698E | 04 |
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| -.19124760E | 04 | -.16968916E | 04 | -.14888228E | 04 | -.12900311E | 04 |
| -.11017854E | 04 | -.92494586E | 03 | -.76003488E | 03 | -.60729976E | 03 |

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| -.46667868E 03 | -.52592114E 03 | -.52249579E 03 | -.47470791E 03 |
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| .56119978E 03  | .58346421E 03  | .59727043E 03  | .60345485E 03  |
| .60274249E 03  | .29826334E 04  | .44114011E 04  | .51276391E 04  |
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| .33643849E 03  | .59617056E 03  | .84319258E 03  | .10640630E 04  |
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| .25788885E 04  | .23590777E 04  | .20442727E 04  | .16715035E 04  |
| .12694119E 04  | .85977893E 03  | .45880471E 03  | .78182044E 02  |
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| -.22583769E 03 | -.65013135E 02 | .65434308E 02  | .16754725E 03  |
| .24362469E 03  | .12315425E 04  | .18280039E 04  | .21288304E 04  |
| .22112135E 04  | .21368678E 04  | .19546988E 04  | .17030533E 04  |
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| .80543451E 02  | .10114775E 03  | .12514029E 03  | .15522553E 03  |
| .19281668E 03  | .23836698E 03  | .29163536E 03  | .35189770E 03  |
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| .16269544E 04  | .14086208E 04  | .12195766E 04  | .10610539E 04  |
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| .96945735E 03  | .12096838E 04  | .14178240E 04  | .15925989E 04  |
| .17340524E 04  | .18432079E 04  | .19217907E 04  | .19720065E 04  |
| .19963283E 04  | .68104054E 02  | -.11157102E 04 | -.17446795E 04 |
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| .28825915E 03  | -.20589185E 03 | -.69121496E 03 | -.11538441E 04 |
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| -.28850456E 04 | -.30990723E 04 | -.32706058E 04 | -.34020345E 04 |
| -.34961322E 04 | -.55767947E 04 | -.68253775E 04 | -.74464963E 04 |
| -.76046324E 04 | -.74309897E 04 | -.70292845E 04 | -.64806253E 04 |
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| -.32537191E 04 | -.27015790E 04 | -.22095333E 04 | -.17806359E 04 |
| -.14151495E 04 | -.11112806E 04 | -.86576977E 03 | -.67436435E 03 |
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| .21805001E 04  | .20737389E 04  | .18165330E 04  | .14620171E 04  |
| .10514643E 04  | .61644050E 03  | .18061446E 03  | -.23873006E 03 |
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| .41145166E 03  | .38969813E 03  | .24588769E 03  | .28459250E 02  |
| -.22591354E 03 | -.49005284E 03 | -.74438942E 03 | -.97539780E 03 |
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| -.15854793E 04 | -.15942653E 04 | -.15705563E 04 | -.15180012E 04 |
| -.14405447E 04 | -.19814620E 02 | .89402663E 03  | .14290707E 04  |
| .16874139E 04  | .17497231E 04  | .16790761E 04  | .15241842E 04  |
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| .48977369E 03  | .33655291E 03  | .21573607E 03  | .12805895E 03  |
| .72704453E 02  | .47754740E 02  | .50550377E 02  | .77972428E 02  |
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| -.23354723E 04 | -.15153545E 04 | -.70846877E 03 | .54329615E 02  |
| .75201423E 03  | .13713003E 04  | .19049269E 04  | .23502471E 04  |
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| .19653557E 04  | .20568093E 04  | .20706929E 04  | .20152834E 04  |
| .18994818E 04  | -.54719044E 03 | -.21327178E 04 | -.30757125E 04 |
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| -.12324157E 04 | -.12119406E 04 | -.12351166E 04 | -.12965675E 04 |

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|----------------|----------------|----------------|----------------|
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| -.30571929E 04 | -.30012212E 04 | -.29670357E 04 | -.29519059E 04 |
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| -.83930658E 03 | -.11159208E 04 | -.13837991E 04 | -.16244730E 04 |
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| .31326158E 04  | .31947493E 04  | .32419525E 04  | .32749094E 04  |
| .32943841E 04  | .33011992E 04  | .32962169E 04  | .32803240E 04  |
| .32544163E 04  | .32193886E 04  | .31761232E 04  | .31254826E 04  |
| .30683028E 04  | .30053865E 04  | .29375013E 04  | .28653744E 04  |
| .27896916E 04  | .27110960E 04  | .26301861E 04  | .25475175E 04  |
| .24635491E 04  | .23788597E 04  | .22938197E 04  | .22088163E 04  |
| .21241986E 04  | .20402787E 04  | .19573338E 04  | .18756085E 04  |
| .17953159E 04  | .17166409E 04  | .16397409E 04  | .15647486E 04  |
| .14917740E 04  | .14209053E 04  | .13522122E 04  | .12857460E 04  |
| .12215427E 04  | .11596237E 04  | .10999975E 04  | .10426612E 04  |
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| .78964892E 03  | .74549320E 03  | .70338389E 03  | .66326578E 03  |
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| .29518555E 03  | .27648016E 03  | .25885739E 03  | .24226397E 03  |
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how the actual kernel can be generated by the estimated minimum kernel. Hence the original IRT can be generated by the minimum kernel.

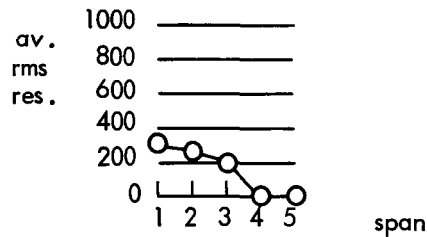


Fig. 29. Average rms residual as a function of duration for the actual kernel of Example 3.

In order to be more convincing yet, the homogeneous equations for the samples of the actual kernel itself were solved for durations 1-5. The plot of the average rms residual is shown in Fig. 29, and indicates 4 to be the "correct" span. The solution

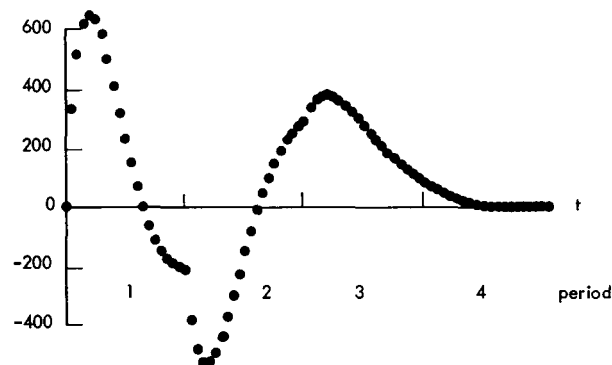


Fig. 30. Solutions to the homogeneous equations for the actual kernel of Example 3.

for duration 4 is plotted in Fig. 30, and is essentially the same as that previously obtained from the original IRT.

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